Economics Prelims Probability & Statistics

Hilary Term Collection

January 2025

Instructions: You have 90 minutes to complete this closed-book, closed-notes collection. Graphing/programmable calculators are not permitted, but scientific or ordinary four-function calculators are allowed. Show your work for full credit. A table of the standard normal distribution is attached to this document.

Question:	1	2	3	4	5	6	7	Total
Points:	5	10	10	10	5	30	30	100
Score:								

☺☺☺ **Good Luck!** ☺☺☺

Part A

Answer each of the following short questions, many of which are taken *verbatim* from your review questions.

[5] 1. Let X be a random variable with support set $\{-1, 0, +1\}$, $p(-1) = 1/4$ and $p(0) = 1/2$. Write down the CDF of X.

2. Let X and Y be random variables and a, b, c be constants. Use the linearity of expectation to derive the following results

[5] (a) $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

[5] (b) $\text{Var}(a + bX + cY) = b^2\text{Var}(X) + c^2\text{Var}(Y) + 2bc\text{Cov}(X, Y)$

To answer the next question you will need to consult the normal tables appended to this document.

3. Let $X \sim \text{N}(0, 1)$. Calculate the following:

[2] (a) $\mathbb{P}(X = 0)$

[2] (b) $\mathbb{P}(X \leq 1)$

[3] (c) $\mathbb{P}(-1.2 \leq X \leq 1.2)$

[3] (d) The value of t such that $\mathbb{P}(X > t) = 0.8$

4. Let X be a Uniform$(0, 1)$ random variable.

[5] (a) Calculate $\mathbb{E}(X)$.

[5] (b) Calculate $\mathbb{E}(X^2)$

[5] 5. Ruth places one ball into her bag in secret: it is red with probability r and blue with probability $(1 - r)$. You watch her place a second ball in the bag: this second ball is red. You shake the bag, reach in, and pull out a ball. The ball you draw is red. Based on this information, calculate the probability that the other ball, the one still in the bag, is *also red*. Express your answer in terms of r.

Part B

Answer both of these questions. They are more challenging than those from Part A, and ask you to apply what you've learned to more complicated situations.

6. Let X_1 and X_2 be independent Bernoulli random variables, each with probability of success equal to p and define the random variable Y as follows:

$$Y = \begin{cases} -1 & \text{if } X_1 = X_2 \\ 0 & \text{if } X_1 = 1 \text{ and } X_2 = 0 \\ 1 & \text{if } X_1 = 0 \text{ and } X_2 = 1 \end{cases}$$

[5] (a) Write out the joint pmf of (X_1, X_2) in a 2×2 table with values of X_1 in the rows and values of X_2 in the columns.

[5] (b) Using your answer to the preceding part, calculate the marginal pmf of Y.

[5] (c) Calculate $\mathbb{P}(Y = 1|Y \neq -1)$.

[5] (d) Frank and Ruth are playing tennis and need a fair way to decide who serves first. They decide to flip a coin, but unfortunately the only one they can find is *biased*: it comes up heads with probability $p \neq 1/2$. Based on your results from the preceding parts, suggest a way that Frank and Ruth can *simulate* a fair coin toss by tossing their biased coin *more than once*.

[10] (e) Fun trivia fact: the method you proposed in the preceding part is often called the *von Neumann extractor*, which sounds extremely bad-ass. If Frank and Ruth use this method, how many times will they need to flip their unfair coin–on average–to determine which of them will serve first? Explain how your result depends on p and why. Even if you can't calculate the expectation exactly, try to say as much as you can about it.

7. Let X be a continuous RV with support $[0, 1]$ and density function $f_X(x) = 2x$ for $x \in [0, 1]$, zero otherwise.

[5] (a) Find the CDF F_X of X.

[5] (b) Calculate $\mu_X \equiv \mathbb{E}(X)$.

You are bidding on a government contract that has a value of v to your firm. Suppose that v is measured in "millions of dollars" and $0 \leq v \leq 1$. Note that v is a known constant from your perspective. If you bid b and your bid is accepted, your firm earns a profit of $\pi = v - b$. Other firms will also bid on this contract, and their values may not equals yours Each firm submits its bid in a sealed envelope. The highest bidder wins the contract and pays their bid; no other bidders make payments.

Suppose that you always bid to maximize your firm's *expected profit*. To choose the optimal bid $b^*(v)$ as a function of your value v, you need to consider the *maximum bid* $B_{\max}$ that your competitors will submit. From your perspective this is a random variable, since you are not party to other firms' cost structures. Based on your research you conclude that $B_{\max}$ has the same CDF as X from above.

[5] (c) Derive an expression for your expected profit $\mathbb{E}[\pi(b)]$ as a function of your bid b and your value v.

[5] (d) Using the result of the preceding part, derive a function for your optimal bid $b^*(v)$ as a function of your value v.

[10] (e) Now suppose that the rules of the auction change. Bids are still submitted in sealed envelopes, the highest bid wins, and only the winner pays. But if you win the auction you pay the bid of the *second highest bidder* rather than your own bid. In other words, if your firm wins you pay $B_{\max}$. Would your optimal bidding rule change? Explain.