Prelims Probability & Statistics

Hilary Term Collection

January 2024

Instructions: You have 90 minutes to complete this closed-book, closed-notes collection. Graphing/programmable calculators are not permitted, but scientific or ordinary four-function calculators are allowed. Show your work for full credit. A table of the standard normal distribution is attached to this document.

Question:	1	2	3	4	5	6	7	8	Total
Points:	5	5	5	5	10	10	30	30	100
Score:									

☺☺☺ **Good Luck!** ☺☺☺

Part A

Answer each of the following short questions, nearly all of which are taken *verbatim* from your assigned review and problem set questions.

[5] 1. Let $X \sim \mathrm{N}(\mu = -2, \sigma^2 = 25)$. Without consulting a table, what is the approximate value of $\mathbb{P}(-12 \leq X \leq 8)$?

[5] 2. Let X be a discrete random variable that is equally likely to take on the values 2 and 4 and never takes on any other values. Prove or disprove the following claims, either by using a property of expected value that you have learned, or by direct calculation

(a) $\mathbb{E}(X + 10) = \mathbb{E}(X) + 10$

(b) $\mathbb{E}(X/10) = \mathbb{E}(X)/10$

(c) $\mathbb{E}(10/X) = 10/\mathbb{E}(X)$

(d) $\mathbb{E}(X^2) = \mathbb{E}[(X)]^2$

(e) $\mathbb{E}\left(\dfrac{5X+2}{10}\right) = \dfrac{5\mathbb{E}(X)+2}{10}$

[5] 3. Let X and Y be independent random variables with $X \in \{-1, 1\}$ and $p_X(1) = 0.5$ while $Y \in \{0, 1\}$ and $p_Y(1) = 0.8$. Write out the joint pmf p_{XY} of (X, Y) in tabular form, with X in the rows of the table and Y in the columns.

[5] 4. Let X and Y be random variables such that $\mathbb{E}[Y] \neq 0$. Show that

$$\frac{\mathbb{E}\,[X]}{\mathbb{E}\,[Y]} - \mathbb{E}\left[\frac{X}{Y}\right] = \frac{\mathrm{Cov}(X/Y, Y)}{\mathbb{E}\,[Y]}.$$

[10] 5. Let A and B be two events. Use the axioms of probability to show that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

[10] 6. I flip 20 fair coins, and discard any that come up heads. I then flip the remaining coins a second time and count up the number of tails T that I obtain. Use the linearity of expectation to calculate $\mathbb{E}(T)$.

Part B

Answer both of these questions. They are somewhat more challenging than those from Part A, and ask you to apply what you've learned to more complicated situations.

7. Stephen has two urns, each of which contains three balls. The "blue urn" contains two blue balls and one red ball; the "red urn" contains two red balls and one blue ball. Neither urn has any distinguishing characteristics. Without looking inside to see which balls they contain, there is no way to tell the urns apart. To determine whether Frank, Natalie and Ruth should be allowed to continue teaching undergraduates, Stephen decides to assess their understanding of conditional probability. He flips a fair coin to choose randomly between the two urns. Then he lets Frank, Natalie and Ruth make one draw from the urn apiece. They're not allowed to peek inside the urn, and they have to **replace their ball** after making the draw. They're also not allowed to show each other which ball they drew. After each tutor makes their draw, they state aloud which urn they think Stephen has chosen.

[5] (a) Frank goes first: he draws a blue ball so he guesses that Stephen has chosen the blue urn. Based on the information available to him, what is the probability that Frank's guess is correct?

[10] (b) Natalie goes second. She knows that Frank is rational: since he guessed "blue urn" this means he must have seen a blue ball. She reaches into the urn and draws a red ball. Show that, based on all the information available to Natalie, the probability that Stephen chose the blue urn is 1/2.

[10] (c) Natalie guesses "blue urn." Ruth knows that both Frank and Natalie are rational. Since Frank guessed "blue urn", he must have drawn a blue ball. But she can't be sure which color ball Natalie drew. Natalie always guesses the urn that has the higher probability based on the information available to her. In the event of a tie, she guesses randomly. Since Natalie guessed "blue urn" Ruth realizes that there are two possibilities: either Natalie drew a blue ball, or she drew a red ball and randomly guessed "blue urn." Ruth reaches into the urn and draws a red ball. What should Ruth guess? Find the probability that her guess is correct, based on the information available to her.

[5] (d) Now suppose that Frank, Natalie, and Ruth had some way to surreptitiously inform one another about which balls each of them had drawn, without Stephen realizing. As before, Frank draws a blue, Natalie red, and Ruth red. But now Ruth knows exactly which ball each of them drew. What should Ruth guess? Find the probability that her guess is correct based on the information available to her in this alternative scenario. **Note that you can solve this part *without* solving part (c).**

8. Sam and Caroline have \$100 and need to make big returns *fast*. Elon offers them the chance to play the following game. First Sam and Caroline must choose the fraction φ of the \$100 that they want to bet. This can be any real number between 0 and 1 inclusive, so they can bet *nothing*, *everything*, or anything in between. Then Elon makes a random draw X_1 with probability density function $f(x) = 2x$ for $x \in [0, 1]$, zero otherwise. If $X_1 > 1/2$, Sam and Caroline win: they keep their initial \$100 and Elon gives them an additional $\varphi \times \$100$. If $X_1 \leq 1/2$, Sam and Caroline lose: they have to pay $\varphi \times \$100$ to Elon. The game then proceeds to *round two*: Elon makes a *second* random draw X_2 that is independent of X_1 but has the same pdf. The rules in round two are the same as before, except that Sam and Caroline now bet a fraction φ of however much money they have *after* round one. Depending on whether they win or lose in the first round, this could be more or less than \$100. The game ends after round two. Note that the same value of φ is used in each round.

[5] (a) Calculate $p \equiv \mathbb{P}(X_1 > 1/2) = \mathbb{P}(X_2 > 1/2)$.

[5] (b) Let R_1 be a random variable that takes the value 1 if $X_1 > 1/2$ and 0 otherwise. Define R_2 analogously for X_2. Write down the probability mass function of the random variable $S \equiv R_1 + R_2$.

[5] (c) Sam wants to choose $\varphi = 1$ and risk *everything* in both rounds of the game. In expectation, how much money will Sam and Caroline have at the end of the game if they follow this plan?

[5] (d) Caroline thinks Sam is crazy: setting $\varphi = 1$ means there's a good chance they'll go bust, ending up with \$0 at the end of the game. Caroline proposes setting $\varphi = 1/2$ instead. If they follow Caroline's strategy, what how much money will they have in expectation at the end of the game? What is the chance of going bust under Caroline's strategy?

[10] (e) Let $W(\varphi)$ be the amount of money that Sam and Caroline have at the end of the game as a function of the fraction φ that they choose to bet. It turns out that $W(\varphi) = \$100 \times (1+\varphi)^S(1-\varphi)^{2-S}$. Briefly explain why this is so, and then use this expression to show that $\varphi = 1/2$ maximizes $\mathbb{E}[\log W(f)]$, where log denotes the natural logarithm. You don't have to check the second-order condition.