Prelims Probability & Statistics

Hilary Term Collection

January 12th, 2023

Instructions: You have 90 minutes to complete this closed-book, closed-notes collection. Graphing/programmable calculators are not permitted, but scientific or ordinary four-function calculators are allowed. Show your work for full credit. A table of the standard normal distribution is attached to this document.

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

☺☺☺ **Good Luck!** ☺☺☺

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1. All parts of this question are taken *verbatim* from your assigned practice problems.

[2] (a) Let X be a variable with support set $\{-1, +1\}$ and $p(-1) = 1/3$. Write down the CDF of X.

[2] (b) If X is a continuous random variable and a, b are constants, how do we calculate $\mathbb{P}(a \leq X \leq b)$?

[2] (c) Let A and B be two mutually exclusive events, each with positive probability. Are they independent? Explain briefly.

[2] (d) Let X and Y be random variables such that $\text{Var}(X) = 2$, $\text{Var}(Y) = 1$, and $\text{Cov}(X, Y) = 0$. Calculate $\text{Var}(X - Y)$.

[5] (e) Let X be a discrete random variable that is equally likely to take on the values 2 and 4 and never takes on any other values. Prove or disprove the following claims, either by using a property of expected value that you have learned, or by direct calculation:

(i) $E(5X + 2)/10 = [5E(X) + 2]/10$

(ii) $E(10/X) = 10/E(X)$

(iii) $E(X^2) = [E(X)]^2$

[6] (f) Let X and U be random variables with $\mathbb{E}(U|X) = 0$ and $\mathbb{E}(X) = 5$. Let a and b be constants, and further define $Y = a + bX + U$.

(i) Show that $\mathbb{E}(U) = 0$.

(ii) Calculate $\mathbb{E}(Y|X = 0)$.

(iii) Suppose that $\mathbb{E}(U) = 0$ but $\mathbb{E}(U|X)$ is unknown. Would this make any difference to your answers in parts (b) and (c) above?

[6] (g) Let $X \sim N(0, 1)$. Calculate the following quantities, using the standard normal table at the end of this document if needed.

(i) $\mathbb{P}(X = 0)$

(ii) $\mathbb{P}(X \leq -1)$

(iii) $\mathbb{P}(-1.2 \leq X \leq 1.2)$

(iv) the value of t such that $\mathbb{P}(X \leq t) = 0.5$

(v) the value of t such that $\mathbb{P}(X > t) = 0.8$

(vi) the value of $t > 0$ such that $\mathbb{P}(-t \leq X \leq t) = 0.9$

2. In the land of Wendimoor, one in fifty people have a disease called *pararibulitis* that causes painful and disturbing hallucinations. Fortunately this condition is easily treated by a daily dose of *Subtracterall*. Because Subtracterall has some mildly unpleasant side effects, the physicians of Wendimoor only prescribe it for patients who test positive for pararibulitis. The test has an accuracy of 80%. In other words, a person with pararibulitis has an 80% of testing positive and a person without pararibulitis has an 80% chance of testing negative.

[15] (a) Amanda is a randomly-chosen citizen of Wendimoor. We test her for *pararibulitis* and the test comes back positive. What is the probability that she has the disease? In your solution, let T be the event that Amanda **tests** positive and D be the event that she has the **disease**.

[6] (b) Half of people with pararibulitis notice that they have symptoms and visit a physician to get tested. At the same time, one in ten of people who *do not* have pararibulitis *mistakenly believe* that they have symptoms and visit a physician to get tested. Anyone who visits a physician and tests positive gets a prescription; no one else does. Among people with pararibulitis, what fraction have a prescription? Among people who *do not* have pararibulitis, what fraction have a prescription? Let S be the event that someone is prescribed **Subtracterall** and V be the even that she **visits** a physician to get tested.

[4] (c) Continuing from part (b), Amanda argues that pararibulitis is *under-diagnosed*: most people who have the disease are not prescribed the Subtracterall. Todd argues that pararibulitis is *over-diagnosed*: most people with a prescription for Subtracterall do not in fact have pararibulitis. Who's right? Explain briefly.

3. Let $X_1 \sim \text{Bernoulli}(1/2)$ independently of $X_2 \sim \text{Bernoulli}(2/3)$ and define $Y = 2X_1$ and $Z = X_2 - X_1$.

[15] (a) Express the joint pmf of Y and Z in tabular form. Please put the realizations of Y in the *rows* of the table and realizations of Z in the *columns*.

[3] (b) What is the marginal pmf of Y?

[3] (c) What is the marginal pmf of Z?

[4] (d) Calculate the covariance between Y and Z. Are Y and Z independent?

4. Let Y be a continuous random variable with support $[0, 1]$ and pdf $f(y) = Cy^3(1-y)$.

[6] (a) Calculate the value of the constant C in the pdf of Y.

[7] (b) Calculate the CDF of Y.

[6] (c) Calculate the expected value of Y.

[6] (d) Calculate the variance of Y.