Bayesian Double Machine Learning for Causal Inference

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My Research Interests

Econometrics

Causal Inference, Spillovers, Measurement Error, Model Selection, Bayesian Inference

Applied Work

Childhood Lead Exposure, Pawn Lending in Mexico City, Colombian Civil Conflict

Overview of Today's Talk

- Causal inference is hard, especially when there are many controls.
- Bayesian approach is appealing, but doesn't work out-of-the-box
- Find a way to combine the advantages of Bayes with good Frequentist properties (bias / variance / coverage probability)
- Related to Frequentist literature on "Double Machine Learning" but aims to improve on finite-sample performance.
- Workshop on Bayesian Causal Inference this Friday: email me for a link!

The Problem / Model

$$Y_i = \alpha D_i + X_i'\beta + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | D_i, X_i] = 0, \quad i = 1, \ldots, n$$

- Learn effect α of treatment D_i (not necessarily binary)
- Selection-on-observables: p-vector of controls X_i
- OLS: unbiased and consistent estimator of α, but noisy if p is large
- Drop control $X_i^{(j)}$ that is correlated with $D_i \Rightarrow$ biased estimate of α if $\beta^{(j)} \neq 0$.

Naïve Shrinkage Estimator: Ridge Regression

Assume everything de-meaned, X scale-normalized

Unique, closed-form solution even if $p > n$

$$\begin{bmatrix} \widehat{\alpha}_{\text{naive}} \\ \widehat{\beta}_{\text{naive}} \end{bmatrix} = \left[\begin{pmatrix} D'D & D'X \\ X'D & X'X \end{pmatrix} + \begin{pmatrix} 0 & 0_p' \\ 0_p & \lambda \mathbb{I}_p \end{pmatrix} \right]^{-1} \begin{pmatrix} D'Y \\ X'Y \end{pmatrix}, \quad \lambda \equiv \frac{\sigma_\varepsilon^2}{\sigma_\beta^2}.$$

Frequentist Interpretation

Minimize $(Y - \alpha D - X\beta)'(Y - \alpha D - X\beta) + \lambda \beta' \beta$

Bayesian Interpretation

Posterior mean: σ_ε known, flat prior on α, independent Normal$(0, \sigma_\beta^2)$ priors on β_j

Regularization-Induced Confounding (RIC)

Term coined by Hahn et al. (2018)

If $\lambda > 0$, bias from correlation between D and residuals:

$$\text{Bias}(\widehat{\alpha}_{\text{naive}}) = \widehat{\omega}' \left[\mathbb{I}_p - (R + \lambda \mathbb{I}_p)^{-1} R\right] \beta$$

$$\text{Var}(\widehat{\alpha}_{\text{naive}}) = \sigma_\varepsilon^2 \left[(D'D)^{-1} + \widehat{\omega}'(R + \lambda \mathbb{I}_p)^{-1} R (R + \lambda \mathbb{I}_p)^{-1} \widehat{\omega}\right]$$

Notation

$$\widehat{\omega}_j = (D'D)^{-1} D' X_j, \quad \widehat{E}_j = X_j - \widehat{\omega}_j X_j, \quad R = \widehat{E}'\widehat{E}$$

Problem

For $\lambda > 0$, bias depends crucially on $\widehat{\omega}$ and β; strong confounding $\Rightarrow$ large bias

Adding a First-Stage

Just a Projection

$$Y = \alpha D + X'\beta + \varepsilon, \quad \mathbb{E}[\varepsilon|X, D] = 0$$
$$D = X'\gamma + V, \quad \mathbb{E}[V|X] = 0$$

Implied by Casual Assumption

$$\text{Cov}(\varepsilon, V) = \text{Cov}(\varepsilon, D - X'\gamma) = \text{Cov}(\varepsilon, D) - \text{Cov}(\varepsilon, X')\gamma = 0.$$

Idea

Maybe adding this regression allows us to learn the degree of counfounding.

Adding the D on X regression has no effect!

"Bayes Ignorability" – Linero (2023; JASA)

Bayes' Theorem

$$\pi(\theta|Y, D, X) \propto f(Y, D|X, \theta) \times \pi(\theta)$$

$\text{Cov}(\varepsilon, V) = 0 \Rightarrow$ no common parameters!

$$f(Y, D|X, \theta) = f(Y|D, X, \theta)f(D|X, \theta) = f(Y|D, X, \alpha, \beta, \sigma_\varepsilon^2) \times f(D|X, \gamma, \sigma_V^2)$$

Problem

Unless prior treats β and γ as dependent, adding the D on X regression has no effect!

Our Solution: Bayesian Double Machine Learning (BDML)

From Structural to Reduced Form

$$Y_i = \alpha D_i + X_i'\beta + \varepsilon_i = X_i'(\alpha\gamma + \beta) + (\varepsilon_i + \alpha V_i) = X_i'\delta + U_i$$

$$\begin{aligned} Y_i &= X_i'\delta + U_i \\ D_i &= X_i'\gamma + V_i \end{aligned} \qquad \begin{bmatrix} U_i \\ V_i \end{bmatrix} \Bigg| X_i \sim \text{Normal}_2(0, \Sigma), \quad \Sigma = \begin{bmatrix} \sigma_\varepsilon^2 + \alpha^2\sigma_V^2 & \alpha\sigma_V^2 \\ \alpha\sigma_V^2 & \sigma_V^2 \end{bmatrix}$$

BDML Algorithm

1. Place "standard" priors on reduced form parameters (δ, γ, Σ)
2. Draw from posterior $(\delta, \gamma, \Sigma)|(X, D, Y)$
3. Posterior draws for $\Sigma \implies$ posterior draws for $\alpha = \sigma_{UV}/\sigma_V^2$

BDML versus Frequentist Double Machine Learning (FDML)

e.g. Chernozhukov et al. (2018; Econometrics J.)

FDML Optimizes

Plug in "Machine Learning" estimators of reduced form parameters: $(\widehat{\delta}_{\text{ML}}, \widehat{\gamma}_{\text{ML}})$

$$\widehat{\alpha}_{\text{FDML}} = \frac{\sum_{i=1}^{n}(Y_i - X_i'\widehat{\delta}_{\text{ML}})(D_i - X_i'\widehat{\gamma}_{\text{ML}})}{\sum_{i=1}^{n}(D_i - X_i'\widehat{\gamma}_{\text{ML}})^2}.$$

BDML Marginalizes

Posterior for α averages over posterior uncertainty about γ and δ

Theoretical Results

$$Y_i = X_i'\delta + U_i \qquad D_i = X_i'\gamma + V_i$$

$$\begin{bmatrix} U_i \\ V_i \end{bmatrix} \Bigg| X_i \sim \text{Normal}_2(0, \Sigma)$$

$$\pi(\Sigma, \delta, \gamma) \propto \pi(\Sigma)\pi(\delta)\pi(\gamma)$$

$$\Sigma \sim \text{Inverse-Wishart}(\nu_0, \Sigma_0)$$

$$\delta \sim \text{Normal}_p(0, \mathbb{I}_p/\tau_\delta)$$

$$\gamma \sim \text{Normal}_p(0, \mathbb{I}_p/\tau_\gamma)$$

Naïve Approach

Analogous but with single structural equation and $\beta \sim \text{Normal}(0, \mathbb{I}_p/\tau_\beta)$

Asymptotic Framework

Fixed true parameters $(\Sigma^*, \delta^*, \gamma^*)$; $n \to \infty$ (large sample); $p \to \infty$ (many controls)

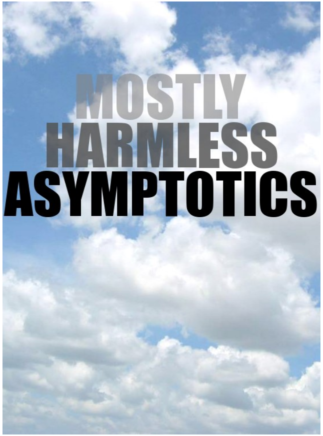
Our asymptotic framework ensures bounded R-squared.

Rate Restrictions

(i) sample size dominates # of controls: $p/n \to 0$

(ii) sample size dominate prior precisions: $\tau/n \to 0$

(iii) precisions of same order as # controls: $\tau \asymp p$



Regularity Conditions

(i) $p < n$

(ii) $\text{Var}(X) \equiv \Sigma_X$ "well-behaved" as $p \to \infty$

(iii) $\lim_{p\to\infty} \sum_{j=1}^{p} (\delta_j^*)^2 < \infty, \quad \lim_{p\to\infty} \sum_{j=1}^{p} (\gamma_j^*)^2 < \infty$

(iv) iid errors/controls, $\mathbb{E}(X_i) = 0$, finite & p.d. Σ^*

Selection Bias in the Limit

When p and n are large, what are our implied beliefs about selection bias?

$$\text{SB} \equiv [\mathbb{E}(Y_i|D_i = 1) - \mathbb{E}(Y_i|D_i = 0)] - \alpha = [\mathbb{E}(X_i|D_i = 1) - \mathbb{E}(X_i|D_i = 0)]'\beta$$

Naïve Model

Degenerate prior centered at zero: $\text{SB} = \dfrac{\gamma'\Sigma_X\beta}{\sigma_V^2 + \gamma'\Sigma_X\gamma} \to_p 0$

BDML

Non-degenerate prior centered at zero: $\text{SB} \to_p \dfrac{\sigma_{UV}}{\sigma_V^2 + \gamma'\Sigma_X\gamma}$

Summary of Asymptotic Results

Consistency

Naïve, BDML and FDML all provide consistent estimators of α.

Asymptotic Bias

BDML and FDML have bias of order p^2/n^2 compared to p/n for Naïve.

$\sqrt{n}$-Consistency

Naïve requires $p/\sqrt{n} \to 0$; BDML and FDML require only $p/n^{3/4} \to 0$.

Why do we focus on variance?

Bias dominates: if $p/\sqrt{n} \to 0$, all three have the same AVAR.

Simulation Experiment

$$Y_i = \alpha D_i + X_i'\beta + \varepsilon_i$$
$$D_i = X_i'\gamma + V_i$$

$$\{X_i\}_{i=1}^n \sim \text{iid Normal}_p(0, \mathbb{I}_p)$$
$$\{(\varepsilon_i, V_i)\}_{i=1}^n \mid X \sim \text{iid Normal}_2\left(0, \text{diag}\left\{\sigma_\varepsilon^2, 1\right\}\right)$$
$$\beta \mid (X, \varepsilon, V) \sim \text{Normal}_p\left(\mu_\beta, \sigma_\beta^2 \mathbb{I}\right).$$

Linero's (2023) "Fixed" Design

$$\alpha = 2, \quad \gamma = \iota_p/\sqrt{p}, \quad \mu_\beta = -\gamma/2, \quad \sigma_\beta^2 = 1/p, \quad n = 200, \quad p = 100$$

Two Versions of BDML

Both Versions

LKJ(4) Prior on Corr(U, V); Independent Cauchy$(0, 2.5)$ priors on SD(U) and SD(V)

Basic Version

Independent Normal$(0, 5^2)$ priors on the elements of δ and γ.

Hierarchical Version

- Independent Normal$(0, \sigma_\delta^2)$ priors on the elements of δ
- Independent Normal$(0, \sigma_\gamma^2)$ priors on the elements of γ
- Independent Inverse-Gamma$(2, 2)$ priors on $\sigma_\delta, \sigma_\gamma$.

Two-Step "Plug-in" Bayesian Approaches

Preliminary Regression

$\widehat{D}_i \equiv X_i' \widehat{\gamma}_{\text{prelim}} \leftarrow$ estimate from Bayesian regression of D on X.

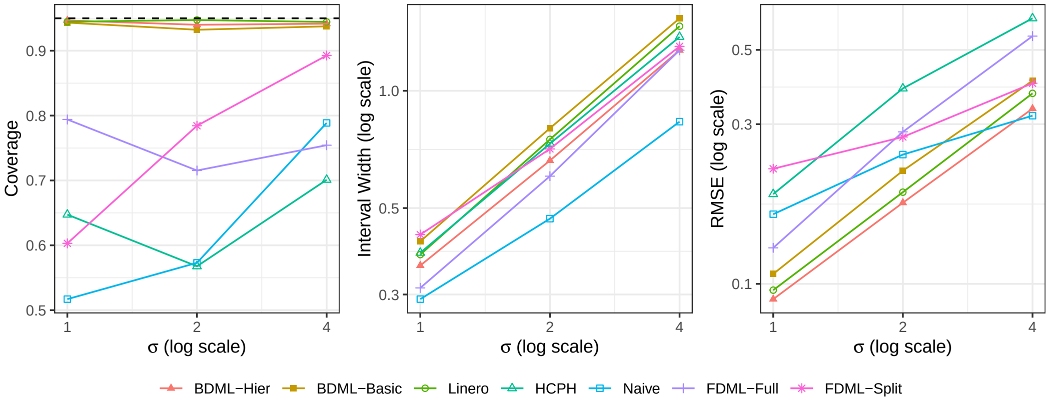
HCPH (Hahn et al, 2018; Bayesian Analysis)

1. Bayesian linear regression of Y on $(D - \widehat{D})$ and X
2. Estimation / inference for α from posterior for $(D - \widehat{D})$ coefficient.

Linero (2023; JASA)

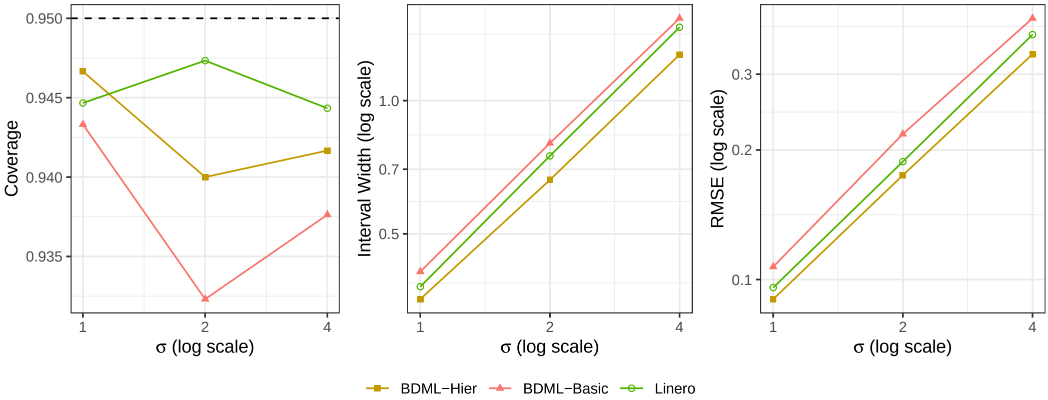
1. Bayesian linear regression of Y on $(D, \widehat{D}, X)$.
2. Estimation / inference for α from posterior the D coefficient.

Simulation Results – 3000 Replications



Only BDML and Linero have correct coverage (Left); Also lower RMSE (Right)

Zooming In: BDML versus Linero



Coverage of Linero & BDML-Hier comparable; BDML-Hier: shortest intervals & lowest RMSE

Thanks for listening!

Summary

- Simple, fully-Bayesian causal inference in a workhorse linear model with many controls.
- Avoids RIC; Excellent Frequentist Properties

In Progress

- More Simulations; Empirical Examples
- Good “default” prior choices?
- Extensions: partially linear model; treatment interactions; instrumental variables?