Local Average Treatment Effects

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Core Empirical Research Methods

Beyond the Textbook IV Model

Heterogenous Treatment Effects

- $Y = \alpha + \beta D + U$ implies that everyone has the same treatment effect: β.
- In reality, treatment effects differ across people.

Local Average Treatment Effects (LATE) Model

- What does IV tell us when treatment effects are heterogeneous?

Binary Treatment and Instrument

$$\beta_{\text{IV}} \equiv \frac{\text{Cov}(Z, Y)}{\text{Cov}(Z, D)} = \frac{\frac{\text{Cov}(Y,Z)}{\text{Var}(Z)}}{\frac{\text{Cov}(D,Z)}{\text{Var}(Z)}} = \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[D|Z=1] - \mathbb{E}[D|Z=0]} \equiv \text{Wald Estimand}$$

Intent-to-treat Effect: $\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]$ (ITT)

- E.g. randomized experiment with treatment **offer** Z and treatment **take-up** D
- **Non-compliance** / randomized encouragement design: D may not equal Z
- In this setting the ITT is the ATE of *offering* treatment.

The Wald Estimand

- ITT is "diluted" by people who are offered $(Z = 1)$ but do not take up $(D = 0)$
- Divide ATE of offer on outcome $Z \rightarrow Y$ by that of offer on take-up $Z \rightarrow D$.
- **Under what assumptions does this give us a meaningful causal quantity?**

Decomposing the ITT Effect

- Example: moving to opportunity (MTO) experiment randomly offered housing vouchers to encourage families to move to a more affluent neighborhood.
- 50% of offered families ($Z = 1$) moved; 20% of non-offered families ($Z = 0$) moved

$$Y = (1 - D)Y_0 + DY_1, \quad p_z \equiv \mathbb{P}(D = 1|Z = z)$$

- $\mathbb{E}[Y|Z = 1]$ is a *mixture* of Y_0 and Y_1 for different types of families:

$$\mathbb{E}[Y|Z = 1] = \underbrace{(1 - p_1)}_{\approx 0.5}\mathbb{E}\left[Y_0|Z = 1, D = 0\right] + \underbrace{p_1}_{\approx 0.5}\mathbb{E}\left[Y_1|Z = 1, D = 1\right]$$

- $\mathbb{E}[Y|Z = 0]$ is a *mixture* of Y_0 and Y_1 for different types of families:

$$\mathbb{E}[Y|Z = 0] = \underbrace{(1 - p_0)}_{\approx 0.8}\mathbb{E}\left[Y_0|Z = 0, D = 0\right] + \underbrace{p_0}_{\approx 0.2}\mathbb{E}\left[Y_1|Z = 0, D = 1\right]$$

Compliance "Types" in the LATE Model

- Catalogue all possible treatment take-up "decision rules"

$$\begin{aligned} \text{Never-taker:} \quad & T = n && \iff && D(Z) = 0 \\ \text{Always-taker:} \quad & T = a && \iff && D(Z) = 1 \\ \text{Complier:} \quad & T = c && \iff && D(Z) = Z \\ \text{Defier:} \quad & T = d && \iff && D(Z) = (1 - Z). \end{aligned}$$

In the MTO Example

- Never-takers: families that refuse to move with or without a voucher
- Always-takers: families that will move with or without a voucher
- Compliers are families that will **only move if given a voucher**
- Defiers are families that will only move if **not** given a voucher

Assumption 1 - Unconfounded Type

For all compliance types $t \in \{a, c, n, d\}$

$$\mathbb{P}(T = t) = \mathbb{P}(T = t | Z = 0) = \mathbb{P}(T = t | Z = 1).$$

Assumption 2 - No Defiers: $\mathbb{P}(T = d) = 0$

Assumption 3 - Mean Exclusion Restriction

For all compliance types $t \in \{a, c, n, d\}$

$$\mathbb{E}\left[Y_0 | Z = 0, T = t\right] = \mathbb{E}\left[Y_0 | Z = 1, T = t\right] = \mathbb{E}\left[Y_0 | T = t\right]$$
$$\mathbb{E}\left[Y_1 | Z = 0, T = t\right] = \mathbb{E}\left[Y_1 | Z = 1, T = t\right] = \mathbb{E}\left[Y_1 | T = t\right]$$

Assumption 4 - Existence of Compliers: $\mathbb{P}(T = c) > 0$

Lemma 1: Assumptions 1–2 $\Longrightarrow$

$$\begin{aligned}
\mathbb{P}(D=1|Z=1) &= \mathbb{P}(T=a)+\mathbb{P}(T=c)\\
\mathbb{P}(D=0|Z=1) &= \mathbb{P}(T=n)\\
\mathbb{P}(D=1|Z=0) &= \mathbb{P}(T=a)\\
\mathbb{P}(D=0|Z=0) &= \mathbb{P}(T=c)+\mathbb{P}(T=n)
\end{aligned}$$

Lemma 2: Assumptions 1–3 $\Longrightarrow$

$$\begin{aligned}
\mathbb{E}[Y|D=1,Z=1] &= \frac{\mathbb{P}(T=a)\mathbb{E}[Y_1|T=a]+\mathbb{P}(T=c)\mathbb{E}[Y_1|T=c]}{\mathbb{P}(T=a)+\mathbb{P}(T=c)}\\
\mathbb{E}[Y|D=0,Z=1] &= \mathbb{E}[Y_0|T=n]\\
\mathbb{E}[Y|D=1,Z=0] &= \mathbb{E}[Y_1|T=a]\\
\mathbb{E}[Y|D=0,Z=0] &= \frac{\mathbb{P}(T=n)\mathbb{E}[Y_0|T=n]+\mathbb{P}(T=c)\mathbb{E}[Y_0|T=c]}{\mathbb{P}(T=n)+\mathbb{P}(T=c)}
\end{aligned}$$

The LATE Theorem: Wald = ATE for Compliers

Theorem: Assumptions 1–4 $\implies$

$$\frac{\mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0)}{\mathbb{E}(D|Z=1)-\mathbb{E}(D|Z=0)} = \mathbb{E}\left[Y_1 - Y_0 | T = c\right]$$

MTO Example

- ITT is the average treatment effect of *offering* a housing voucher.
- Wald = LATE is the average treatment effect of *moving to opportunity* for families who can be induced to move by the voucher from the experiment.

LATE Derivation - Part 1

By iterated expectations and **Lemma 2**

$$
\begin{aligned}
\mathbb{E}(Y|Z=1) &= \mathbb{E}(Y|D=0,Z=1)\mathbb{P}(D=0|Z=1) + \mathbb{E}(Y|D=1,Z=1)\mathbb{P}(D=1|Z=1) \\
&= \mathbb{P}(T=n)\mathbb{E}(Y_0|T=n) + [\mathbb{P}(T=a)\mathbb{E}(Y_1|T=a) + \mathbb{P}(T=c)\mathbb{E}(Y_1|T=c)]
\end{aligned}
$$

Analogously for $Z=0$,

$$
\begin{aligned}
\mathbb{E}(Y|Z=0) &= \mathbb{E}(Y|D=0,Z=0)\mathbb{P}(D=0|Z=0) + \mathbb{E}(Y|D=1,Z=0)\mathbb{P}(D=1|Z=0) \\
&= [\mathbb{P}(T=n)\mathbb{E}(Y_0|T=n) + \mathbb{P}(T=c)\mathbb{E}(Y_0|T=c)] + \mathbb{P}(T=a)\mathbb{E}(Y_1|T=a).
\end{aligned}
$$

Subtracting these gives an expression for the ITT:

$$
\mathbb{E}(Y|Z=1) - \mathbb{E}(Y|Z=0) = \mathbb{P}(T=c)\mathbb{E}(Y_1 - Y_0|T=c).
$$

LATE Derivation - Part 2

ITT = Numerator of Wald Estimand:

$$\mathbb{E}(Y|Z=1) - \mathbb{E}(Y|Z=0) = \mathbb{P}(T=c)\mathbb{E}(Y_1 - Y_0|T=c).$$

For the denominator, **Lemma 1** gives

$$\begin{aligned}
\mathbb{E}(D|Z=1) - \mathbb{E}(D|Z=0) &= \mathbb{P}(D=1|Z=1) - \mathbb{P}(D=1|Z=0) \\
&= [\mathbb{P}(T=a) + \mathbb{P}(T=c)] - \mathbb{P}(T=a) \\
&= \mathbb{P}(T=c)
\end{aligned}$$

since D is binary. Dividing, the Wald Estimand equals $\mathbb{E}(Y_1 - Y_0|T=c)$.

Better LATE than nothing?[1]

- If treatment effects are heterogeneous, IV identifies the **LATE**
- **Local Average Treatment Effect**: average treatment effect for compliers.
- But the definition of "complier" depends on the instrument.
- E.g. a $1,000,000 voucher to "move to opportunity" versus a $100 voucher
- We have an ATE for some people, but we don't know who they are.
- Can't point to anyone in the sample and say "that's a complier!"
- My view: LATE is not always a very interesting parameter.
- More interesting if *most people* are compliers or "the instrument is the policy"
- Beyond LATE: **Marginal Treatment Effects**: slides 1, slides 2

[1] For a more positive view, see Imbens (2010).

We can learn the *average* characteristics of compliers.

E.g. let $F = 1$ if female, zero otherwise. By Bayes' Theorem:

$$\mathbb{P}(F = 1 | T = c) = \frac{\mathbb{P}(T = c | F = 1)\mathbb{P}(F = 1)}{\mathbb{P}(T = c)} = \frac{\mathbb{P}(T = c | F = 1)\mathbb{P}(F = 1)}{\mathbb{E}(D|Z = 1) - \mathbb{E}(D|Z = 0)}.$$

If $Z \perp\!\!\!\perp F$ an argument very similar to that for the Wald denominator gives

$$\mathbb{P}(T = c | F = 1) = \mathbb{E}(D|Z = 1, F = 1) - \mathbb{E}(D|Z = 0, F = 1)$$

Combining these:

$$\mathbb{P}(F = 1 | T = c) = \mathbb{P}(F = 1)\left[\frac{\mathbb{E}(D|Z = 1, F = 1) - \mathbb{E}(D|Z = 0, F = 1)}{\mathbb{E}(D|Z = 1) - \mathbb{E}(D|Z = 0)}\right]$$

so we can learn the *fraction* of compliers who are female.

One-sided Non-compliance

No Always-takers: $Z = 0 \implies D = 0$

- E.g. randomized encouragement design; no access to treatment outside experiment.
- Since there are no always-takers, anyone with $D = 1$ is a complier:

$$\frac{\mathbb{E}(Y|Z=1) - \mathbb{E}(Y|Z=0)}{\mathbb{E}(D|Z=1) - 0} = \mathbb{E}(Y_1 - Y_0|T=c) = \mathbb{E}(Y_1 - Y_0|D=1) = \text{TOT}$$

No Never-takers: $Z = 1 \implies D = 1$

- E.g. Butler Act increased minimum UK school-leaving age from 14 to 15 in 1947.[2]
- Since there are no never-takers, anyone with $D = 0$ is a complier:

$$\frac{\mathbb{E}(Y|Z=1) - \mathbb{E}(Y|Z=0)}{1 - \mathbb{E}(D|Z=0)} = \mathbb{E}(Y_1 - Y_0|T=c) = \mathbb{E}(Y_1 - Y_0|D=0) = \text{TUT}$$

[2] See Oreopoulos (2005) for more details.

Which assumptions are testable in the textbook IV model?

Instrument Relevance

- Since D and Z are observed, directly estimate $\text{Cov}(D, Z)$.
- Beware of weak instruments!

Instrument Exogeneity

- Since U is unobserved, can't directly estimate $\text{Cov}(Z, U)$.
- Could we use the IV residuals?

Simulation with a Bad Instrument

It has a direct effect on Y separate from its effect on D!

```r
library(mvtnorm); library(tidyverse); library(broom); library(AER)
set.seed(587103)
n <- 1e5
sims <- rmvnorm(n, sigma = matrix(c(1, 0.5,
                                    0.5, 1), 2, 2, byrow = TRUE))
U <- sims[,1]
V <- sims[,2]
Z <- rbinom(n, size = 1, prob = 0.3)
D <- -0.5 + 0.3 * Z + V
beta <- 0
Y <- 1 + beta * D - Z + U # Instrument isn't excluded!
```

Bad Instrument Is Uncorrelated with IV Residuals!

```r
iv_results <- ivreg(Y ~ D | Z)
tidy(iv_results) |> knitr::kable(digits = 2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	-0.72	0.04	-17.31	0
D	-3.45	0.10	-35.90	0

```r
cov(residuals(iv_results), Z)
```

```
## [1] -1.534378e-16
```

Z Is Uncorrelated with the IV Residuals *By Construction*

- Let U be the **structural error** and V be the **IV residual**: $V \equiv Y - \alpha_{IV} - \beta_{IV} D$.

$$\beta_{IV} = \frac{\mathrm{Cov}(Z, Y)}{\mathrm{Cov}(Z, D)} = \beta + \frac{\mathrm{Cov}(Z, U)}{\mathrm{Cov}(Z, D)}, \quad \alpha_{IV} = \mathbb{E}(Y) - \beta_{IV}\mathbb{E}(D).$$

- $V = U \iff Z$ is exogenous: the only way to obtain $\beta_{IV} = \beta$ and $\alpha_{IV} = \alpha$.

$$\begin{aligned}\mathrm{Cov}(Z, V) = \mathrm{Cov}(Z, Y - \alpha_{IV} - \beta_{IV} D) &= \mathrm{Cov}(Z, Y) - \beta_{IV}\mathrm{Cov}(Z, D)\\ &= \mathrm{Cov}(Z, Y) - \frac{\mathrm{Cov}(Z, Y)}{\mathrm{Cov}(Z, D)}\mathrm{Cov}(Z, D) = 0.\end{aligned}$$

- $\mathrm{Cov}(Z, V) = 0$ *by construction* even when $\mathrm{Cov}(Z, U) \neq 0$

Multiple Instruments and Over-identification

Assumptions

- $Y = \alpha + \beta D + U$
- $\text{Cov}(Z_1, D) \neq 0, \ \text{Cov}(Z_2, D) \neq 0$
- $\text{Cov}(Z_1, U) = \text{Cov}(Z_2, U) = 0$

$$\beta_{IV}^{(1)} \equiv \frac{\text{Cov}(Z_1, Y)}{\text{Cov}(Z_1, D)} = \beta + \frac{\text{Cov}(Z_1, U)}{\text{Cov}(Z_1, D)}$$

$$\beta_{IV}^{(2)} \equiv \frac{\text{Cov}(Z_2, Y)}{\text{Cov}(Z_2, D)} = \beta + \frac{\text{Cov}(Z_2, U)}{\text{Cov}(Z_2, D)}$$

Implications

- Both IVs identify *same* effect: β
- If not, **at least one is endogenous**

$$\beta_{IV}^{(1)} - \beta_{IV}^{(2)} = \frac{\text{Cov}(Z_1, U)}{\text{Cov}(Z_1, D)} - \frac{\text{Cov}(Z_2, U)}{\text{Cov}(Z_2, D)}$$

Over-identifying Restrictions Test

- Test of null that all MCs identify same parameters.
- Fails in a LATE model: different instruments identify different LATEs!

Are the LATE Assumptions Testable?

LATE Assumptions

1. Unconfounded Type
2. No Defiers
3. Mean Exclusion Restriction
4. Existence of Compliers

At Least One is Testable!

- Assumptions 1–3 $\implies \mathbb{P}(T = c) = \mathbb{E}[D|Z = 1] - \mathbb{E}[D|Z = 0]$
- Thus, Assumption 4 is just **instrument relevance**, hence testable.
- What about the others?

Even Nobel Laureates Make Mistakes

Angrist & Imbens (1994)

Part (i) is similar to an exclusion restriction in a regression model. It is not testable and has to be considered on a case by case basis.

Pearl (1995)

exogeneity . . . can be given an empirical test. The test is not guaranteed to detect all violations of exogeneity, but it can, in certain circumstances, screen out very bad would-be instruments.

Testable Implications of LATE assumptions

- Huber & Mellace (2015)
- Kitagawa (2015)
- Mourifié & Wan (2017)

Example: Card (1995)[3]

- $Y = \log(Wage)$, $D =$ College, $Z =$ Live Nearby

```
library(tidyverse); library(wooldridge); library(estimatr)
card1995 <- as_tibble(card) |>
  mutate(Y = lwage,    # log(wage)
         Z = nearc4,   # Live near 4 year college?
         D = 1 * (educ >= 16)) |> # Attend college? (>=16 years educ)
  select(Y, Z, D)

iv <- iv_robust(Y ~ D | Z, card1995)
ols <- lm_robust(Y ~ D, card1995)
```

[3] Using geographic variation in college proximity to estimate the return to schooling

IV Estimate is Implausibly Large

```r
library(modelsummary)
modelsummary(list(OLS = ols, IV = iv), output = 'latex', fmt = 2,
             gof_omit = 'Num.Obs.|R2|R2 Adj.|AIC|BIC|RMSE',
             coef_omit = '(Intercept)')
```

	OLS	IV
D	0.23	2.27
	(0.02)	(0.55)

Remember: this is on the log scale!

Example of the Huber & Mellace (2015) Approach

- Suppose that all of the LATE assumptions hold and define:

$$r \equiv \frac{\mathbb{P}(T = n)}{\mathbb{P}(T = c) + \mathbb{P}(T = n)} = \frac{\mathbb{P}(D = 0|Z = 1)}{\mathbb{P}(D = 0|Z = 0)} \qquad \text{(by Lemma 1)}$$

- Distribution of $Y|(D = 0, Z = 0)$ is a mixture of Y_0 for compliers and never-takers.
- The mixture contains $r \times 100\%$ never-takers and $(1 - r) \times 100\%$ compliers.
- Let's calculate r in the Card (1995) example:

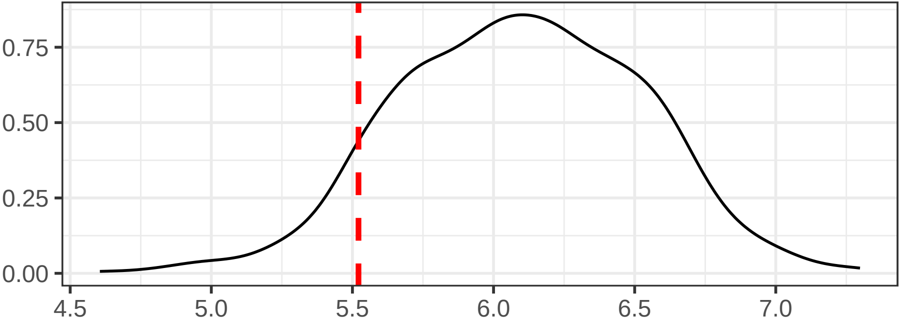
```
r <- card1995 |>
  summarize(p01 = sum(D == 0 & Z == 1) / sum(Z == 1),
            p00 = sum(D == 0 & Z == 0) / sum(Z == 0),
            r = p01 / p00) |> pull(r)
r
```

```
## [1] 0.9115626
```

Density of $Y|(D = 0, Z = 0)$ from Card (1995)

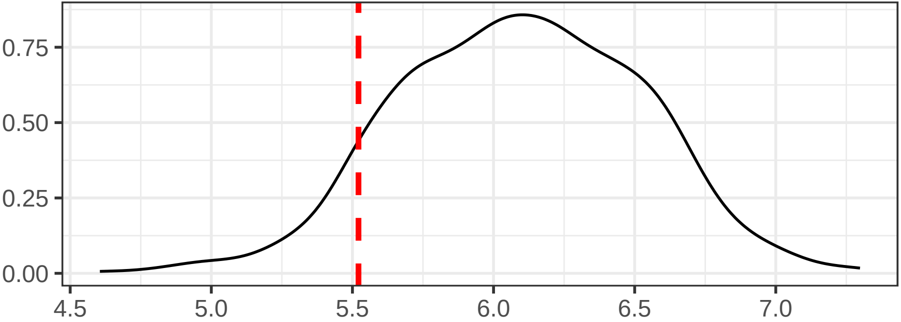
```
y00_density <- card1995 |> filter(D == 0, Z == 0) |>
  ggplot(aes(x = Y)) +
  geom_density() +
  geom_vline(aes(xintercept = quantile(Y, 1 - r)),
             color = 'red', linetype = 'dashed', size = 1) +
  xlab('') + ylab('') +
  theme_bw()
```

Density of $Y|(D = 0, Z = 0)$ from Card (1995)



- This is the density of Y_0 for a mix of never-takers and compliers.
- The mix contains 91% never-takers. But we don't know where they are.
- Dashed red line: 9th %-tile of the density.
- If *all* never-takers are at the top of the distribution, they're above this line.

Density of Y_0 for a mixture containing 91% never-takers, 9% compliers



- If all never-takers are at the top of the distribution, they're above the red line.
- Mean of all observations *above* red line bounds $\mathbb{E}[Y_0|T = n]$ **from above**
- But Lemma 2 shows that $\mathbb{E}(Y_0|T = n) = \mathbb{E}(Y|D = 0, Z = 1)$.
- If this contradicts the upper bound **something must be wrong**.

Contradiction $\implies$ LATE Assumptions Fail

Previous Slide: $\mathbb{E}(Y_0|T=n) \leq \mathbb{E}(Y|D=0, Z=0, Y \geq y_{1-r})$

```
card1995 |>  filter(D == 0, Z == 0) |>
  summarize(ninth_percentile = quantile(Y, 1 - r),
            upper_bound = mean(Y[Y >= ninth_percentile])) |>
  pull(upper_bound)
```

```
## [1] 6.154926
```

Lemma 2: $\mathbb{E}(Y_0|T=n) = \mathbb{E}(Y|D=0, Z=1)$

```
card1995 |> filter(D == 0, Z == 1) |> summarize(mean(Y)) |> pull()
```

```
## [1] 6.254177
```

This contradicts the upper bound! Something must be wrong!