Optional Proofs from Lectures 10 and 11

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Econ 103

None of these proofs is required for this course: I won't test you on them on exams and my upcoming lectures do not assume that you've read this document. However, none of the steps used in any of these proofs goes beyond what we've covered in Econ 103. If you want extra practice working with sums and discrete random variables, going through these proofs will be very helpful. Moreover, if you're interested in going further with probability and statistics, perhaps by doing a minor in statistics at Wharton, the material covered here will help you with the more advanced courses you'll take later in your college career. The proofs I give here are for discrete RVs but you can adapt the same ideas to generalize these results so they cover continuous RVs as well: just replace sums with integrals.

Law of Iterated Expectations

$$\begin{aligned}
E_X\left[E_{Y|X}\left[Y|X\right]\right] &= E_X\left[\sum_y y\; p_{Y|X}(y|x)\right] = \sum_x \left(\sum_y y\; p_{Y|X}(y|x)\right) p_X(x)\\
&= \sum_x \sum_y y\; p_X(x)\; p_{Y|X}(y|x) = \sum_x \sum_y y\; p_{XY}(x,y)\\
&= \sum_y y \sum_x p_{XY}(x,y) = \sum_y y\; p_Y(y) = E[Y]
\end{aligned}$$

The second step is the tricky one. Here what we're doing is calculating the expected value of the function $g(X) = \sum_y p_{Y|X}(y|X)$ with respect to the *marginal* pmf of X.

Independence Implies Zero Covariance

$$
\begin{aligned}
Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= \sum_x \sum_y (x-\mu_X)(y-\mu_Y)p(x,y) \\
&= \sum_x \sum_y (x-\mu_X)(y-\mu_Y)p(x)p(y) \\
&= \sum_x (x-\mu_X)p(x) \left[\sum_y (y-\mu_Y)p(y)\right] \\
&= E[Y-\mu_Y] \sum_x (x-\mu_X)p(x) \\
&= E[Y-\mu_Y]E[X-\mu_X] \\
&= 0
\end{aligned}
$$

Linearity of Expectations

$$
\begin{aligned}
E[aX+bY+c] &= \sum_x \sum_y (ax+by+c)p(x,y) \\
&= \sum_x \sum_y \left[axp(x,y)+byp(x,y)+cp(x,y)\right] \\
&= a\sum_x \sum_y xp(x,y) + b\sum_y \sum_x yp(x,y) + c\sum_y \sum_x p(x,y) \\
&= a\sum_x \sum_y xp(x,y) + b\sum_y \sum_x yp(x,y) + c \\
&= a\sum_x x\left(\sum_y p(x,y)\right) + b\sum_y y\left(\sum_x p(x,y)\right) + c \\
&= a\sum_x xp(x) + b\sum_y yp(y) + c \\
&= aE[X] + bE[Y] + c
\end{aligned}
$$