Finite Measures and Geometric Paradoxes

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December 4, 2006

1 Introduction and Preliminaries

The Axiom of Choice seems like a perfectly natural and uncontroversial assumption, but it is well known to have some unexpected, even counterintuitive implications. In this paper, we use the Axiom of Choice to disprove the existence of certain finite measures, illustrating the inherent limitations of measure theory. Our approach considers decompositions that appear innocuous as they employ only distance preserving maps, but end in paradox.

The proofs in this paper follow Wagon (1993) although all have been expanded and some generalized. We begin by reminding the reader of a few concepts that appear frequently in the discussion to follow.

Definition 1.1 (Group Action)**.** *Let X be a set and G be a group. A binary function $f : G \times X \to X$ is called a group action , or more properly a left group action, if and only if the following properties hold:*

1. $(gh) \cdot x = g \cdot (h \cdot x)$ *for any* $g, h \in G, x \in X$*, and*
2. $e \cdot x = x$ *for any* $x \in X$*, where* e *is the identity element of the group* G*.*

It follows easily from this definition that for any $g \in G$ the function that maps $x \in X$ to $g \cdot x$ is a bijection on X. To show that it is is one-to-one, suppose $g \cdot x_1 = g \cdot x_2$. Since g is an element of the group G, we may left-multiply both sides by the inverse g^{-1} yielding $x_1 = x_2$. To show that this function is onto we must find for some arbitrary $y \in X$ an $x \in X$ such that $g \cdot x = y$. Again, we may left multiply by g^{-1} yielding $x = g^{-1} \cdot y$.

Definition 1.2 (Free Group)**.** *A group G is called a free group if and only if there exists a subset $S \subseteq G$ such that any element $g \in G$ can be written as a unique product (after simplifying) of finitely many elements of S and their inverses.*

It is worth commenting on the uniqueness requirement of the above definition. Note that for $s, t, u \in S$ and $g \in G$, $g = s \cdot t^{-1}$ and $g = s \cdot u \cdot u^{-1} t^{-1}$ are not considered distinct representations of g because one simplifies to the other.

Definition 1.3 (Generating Set of a Group)**.** *For a group G, the subset $S \subseteq G$ such that any element $g \in G$ can be uniquely represented (after simplifying) as a product of finitely many elements of S and their inverses is called the generating set of G. The elements of S are called generators.*

Definition 1.4 (Distance Preserving Map)**.** *Let X and Y be metric spaces with distance functions d_X and d_Y, respectively. We call a map $f : X \to Y$ distance preserving if for any $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$.*

Clearly, a distance preserving map is always an one-to-one. To see why, suppose that $f(x_1) = f(x_2)$ where $x_1, x_2 \in X$ and f is a distance preserving map. Then we have $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. Since d_Y is a metric, and $f(x_1) = f(x_2)$, $d_Y(f(x_1), f(x_2)) = 0$. Hence, $d_X(x_1, x_2) = 0$. Therefore, by the definition of a metric, $x_1 = x_2$. In general, a distance preserving map is not onto. We define this special case as follows:

Definition 1.5 (Isometry)**.** *A bijective, distance preserving map is called an isometry, or more precisely a global isometry.*

2 Paradoxical Decompositions

In this section, we begin our consideration of paradoxical decompositions, first with a definition and then with a series of simple results upon which the remainder of our discussion will be based. We close with a concrete example on the unit circle that has some important measure-theoretic consequences.

Definition 2.1. *Let X be a set, and G a group acting on X. A subset $E \subseteq X$ is called paradoxical with respect to G (or G-paradoxical) if for positive integers m and n there exists a collection of pairwise disjoint subsets*

$A_1, \ldots, A_m, B_1, \ldots, B_n \subseteq E$, and a collection of elements $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that

$$E = \bigcup_{i=1}^{m} g_i A_i = \bigcup_{j=1}^{n} h_j B_j$$

E is called countably G-paradoxical in the case where we have a countably infinite collection of pairwise disjoint subsets $A_1, A_2, \ldots, B_1, B_2, \ldots \subseteq E$, and correspondingly many elements $g_1, g_2, \ldots, h_1, h_2 \ldots \in G$ such that

$$E = \bigcup_{i=1}^{\infty} g_i A_i = \bigcup_{j=1}^{\infty} h_j B_j$$

Theorem 2.1. *A free group G with two generators is paradoxical with respect to G when it acts on itself by left multiplication.*

Proof. Let $S = \{\sigma, \tau\}$ be the generating set of G and let e be the identity element in G. Now denote by $W(\rho)$ the set of all elements $g \in G$ whose (unique) representation begins on the left with ρ, where $\rho \in \{\sigma, \sigma^{-1}, \tau, \tau^{-1}\}$. Then we may clearly express G as the following union of disjoint subsets:

$$G = \{e\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1}).$$

We now construct a paradoxical decomposition of G. To begin, we show that $G \backslash W(\sigma) \subseteq \sigma W(\sigma)$. Suppose $h \in G \backslash W(\sigma)$. Then the representation of h does not begin on the left with σ. Hence, $\sigma^{-1} h \in W(\sigma^{-1})$. Thus, $\sigma(\sigma^{-1} h) \in \sigma W(\sigma^{-1})$, but since $\sigma(\sigma^{-1} h) = h$, we have $h \in \sigma W(\sigma^{-1})$. Because $G \backslash W(\sigma) \subseteq \sigma W(\sigma)$ and $W(\sigma) \cup G \backslash W(\sigma) = G$, we have $G = W(\sigma) \cup \sigma W(\sigma^{-1})$. In complete analogy with the above it can be shown that $G = W(\tau) \cup \tau W(\tau^{-1})$.

Therefore, we have constructed a paradoxical decomposition of G since:

$$\begin{aligned} G &= eW(\tau) \cup \tau W(\tau^{-1}) \\ G &= eW(\sigma) \cup \sigma W(\sigma^{-1}) \end{aligned}$$

and $W(\sigma), W(\sigma^{-1}), W(\tau), W(\tau^{-1})$ are pairwise disjoint by construction. □

Though it may seem trivial, the preceding result will ultimately allow us to construct the famous Hausdorff paradox. However, before we can apply it to more interesting situations, we must generalize our result to group actions by G on sets other than G itself. Fortunately, this is easily accomplished by the following theorem.

Theorem 2.2. *If a group G is paradoxical with respect to itself and acts on a set X with only trivial fixed points, then X is paradoxical with respect to G.*

Proof. Since G is paradoxical with respect to itself, we know that $G = \bigcup_i^m g_i A_i = \bigcup_j^n h_j B_j$ where $A_1, \ldots, A_m, B_1, \ldots, B_n \subseteq G$ are pairwise disjoint and we have $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$. Define an equivalence relation on X as follows: for $x, y \in X$, $x \sim y$ if and only if $Gx = Gy$, where $Gx = \{g \cdot x : g \in G\}$. That is, we call x and y equivalent when they share an orbit. Using the Axiom of Choice, we now select a choice set M containing one representative from each of the resulting equivalence classes.

Note that the collection $\{g(M) : g \in G\}$ forms a partition of X. This is because, given our definition of equivalence, every possible choice set can be generated by left-multiplying our chosen set M by an element of G. This collection will not contain any duplicates because we have assumed that G acts on X with only trivial fixed points (i.e. $g(M) = G$ if and only if g is the identity). Because $\{g(M) : g \in G\}$ partitions X, we clearly have $g_i(M) \cap g_j(M) = \emptyset$ for any $i \neq j$.

Now define $A_i^* = \cup_{i=1}^m \{g(M) : g \in A_i\}$ and $B_j^* = \cup_{j=1}^m \{g(M) : g \in B_j\}$. Because the collection $A_1, \ldots, A_m, B_1, \ldots, B_n$ is pairwise disjoint, the same is true of the collection $A_1^*, \ldots, A_m^*, B_1^*, \ldots, B_n^*$. Thus we have generated a paradox: $X = \bigcup_{i=1}^m g_i A_i^* = \bigcup_{j=1}^n h_j B_j^*$. This follows from our assumption that $G = \bigcup_i^m g_i A_i = \bigcup_j^n h_j B_j$ and the definitions of A_i^*, B_j^*. □

Corollary 2.1. *Any set X is paradoxical with respect to F, where F is a free group with two generators acting on X with only trivial fixed points.*

Proof. This result follows immediately from the preceding two theorems. □

This corollary is in many ways the central result of our discussion. As we shall see, it allows the construction a geometric paradox in $\mathbb{R}^3$. Before concluding this section, we consider a simple example of a countably paradoxical decomposition on the unit circle and explore its measure-theoretic consequences.

Theorem 2.3. *The unit circle, S^1, is countably paradoxical with respect to the group of rotations on the circle, SO_2. Further, the interval $[0,1)$ is countably paradoxical with respect to the group G of translations $mod(1)$.*

Proof. By definition $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The group SO_2 is merely S^1 under multiplication. Let $s_1, s_2 \in S^1$ and define a relation as follows: $s_1 \sim s_2$ if and only if s_1 can be obtained by rotating s_2 by a rational multiple of π radians. Clearly this is an equivalence relation, hence its equivalence classes induce a partition of S^1. By the Axiom of Choice, we may take a representative from each equivalence class. Call this choice set M.

Enumerate the rotations that define equivalence as follows: $\{\rho_i : i \in \mathbb{N}\}$ (this is possible because the rational numbers are countable). Now define by M_i the choice set M rotated through ρ_i radians. Note that the set $\{M_i : i \in \mathbb{N}\}$ partitions S^1. This is because $\rho_i(M)$ generates all possible choice sets for the equivalence relation defined above. Hence, $M_i \cap M_j = \emptyset$ for any $i \neq j$.

By definition, any two of the M_i are congruent. Thus, for any choice of M_i, M_j we can find a ρ_k such that $M_i = \rho_k(M_j)$. In particular, since there are as many even naturals as there are naturals, we can choose $\rho_1', \rho_2', \rho_3', \ldots$ such that $M_1 = \rho_1'(M_2)$, $M_2 = \rho_2'(M_4)$, $M_3 = \rho_3'(M_6)$, etc. In this way, we can generate the entire unit circle by rotating the pairwise disjoint sets M_i where i is even. The same may be done for i odd, although we must choose a different set of rotations. Thus we have:

$$\begin{aligned} S^1 &= \bigcup_{i=1}^{\infty} \rho_i'(M_{2i}) \\ S^1 &= \bigcup_{i=1}^{\infty} \rho_i''(M_{2i-1}) \end{aligned}$$

Therefore, since $M_i \cap M_j = \emptyset$ for any $i \neq j$, S^1 is countably SO_2-paradoxical.

The analogous result for the interval $[0,1)$ and the group of translations modulo 1 is obtained by finding an isomorphism between SO_2 and G. Let $f : [0,1) \to S^1$ be defined by $f(x) = \cos(2\pi x) + \sin(2\pi x)i$. By Euler's formula, $f(x) = e^{2\pi x i}$. This is clearly a bijection, so we need only show that it induces a homomorphism between G and SO_2. Let $x, y \in [0,1)$. Then we have

$$f(x + y \mod 1) = \cos[2\pi(x + y)] + \sin[2\pi(x + y)]i = e^{2\pi(x+y)}$$

The modular arithmetic disappears because sine and cosine are periodic with period 2π. Suppose for example that $x + y \geq 1$. Then $\cos[2\pi(x + y)] = \cos[2\pi + 2\pi(x + y \mod 1)] = \cos[2\pi(x + y \mod 1)]$. The same result clearly holds for sine. Now consider $f(x) \cdot f(y)$. We have

$$\begin{aligned} f(x) \cdot f(y) &= [\cos(2\pi x) + \sin(2\pi x)i][\cos(2\pi y) + \sin(2\pi y)i] \\ &= e^{2\pi x}e^{2\pi y} \\ &= e^{2\pi(x+y)} \end{aligned}$$

Hence our desired result follows. ☐

Corollary 2.2. *There is no finite, rotation-invariant measure that is defined on all subsets of the unit circle. Analogously, there is no finite, translation-invariant measure defined on all subsets of* $\mathbb{R}$.

Proof. Suppose by way of contradiction that μ is in fact a finite, rotation-invariant measure defined on $\mathcal{P}(S^1)$. Let $A = \bigcup_{i=1}^{\infty} M_{2i}$ and $B = \bigcup_{i=1}^{\infty} M_{2i-1}$, where M_i is defined as in the above theorem. Clearly, $A \cup B = S^1$ and $A \cap B = \emptyset$. Since μ is a finite measure, we have $\mu(S^1) = x$ for some $x \in \mathbb{R}_{\geq 0}$. Hence, x=$\mu(S^1) = \mu(A) + \mu(B)$ by σ-additivity. However, we know from the above theorem that A can be rotated to cover all of S^1 as can B. Further, we have assumed that μ is rotation-invariant. Thus, preserving the notation from the above proof, we must have $\mu(\rho_i'(M_{2i})) = \mu(M_{2i})$ and $\mu(\rho_i'(M_{2i-1})) = \mu(M_{2i-1})$ for any $i \in \mathbb{N}$. Substituting and invoking σ-additivity, we have $\mu(A) = \sum_{i=1}^{\infty} \mu(M_{2i}) = \sum_{i=1}^{\infty} \mu(\rho_i'(M_{2i})) = \mu(S^1) = x$ and $\mu(B) = \sum_{i=1}^{\infty} \mu(M_{2i-1}) = \sum_{i=1}^{\infty} \mu(\rho_i'(M_{2i-1})) = \mu(S^1) = x$.

Hence, we have a contradiction since $\mu(A) = \mu(B) = \mu(S^1) = x$, but we showed previously that $x = \mu(S^1) = \mu(A) + \mu(B)$. The isomorphism defined in the previous theorem transfers this result to $\mathbb{R}$. ☐

3 The Hausdorff Paradox

In the previous section we laid the groundwork for paradoxical decompositions on arbitrary spaces, proving that whenever a free group F with two generators acts on a set X with only the trivial fixed point, X is paradoxical with respect to F. We will now use this result to provide a particularly important example. We begin by showing that a group of Euclidean isometries on $\mathbb{R}^3$ contains a free subgroup with two generators, and use this fact to construct the famous Hausdorff paradox. The proof given here, following Wagon (1993), differs somewhat from Hausdorff's original formulation (1914). The main ideas, however, are similar.

Theorem 3.1. *The rotation group SO_3 contains a free subgroup with two generators.*

Proof. The group SO_3 is simply the set of orthogonal matrices with determinant 1 under the operation of matrix multiplication (recall that a matrix is called orthogonal if its transpose equals its inverse). Define $\phi^{\pm 1}, \rho^{\pm 1} \in SO_3$ as follows

$$\phi^{\pm 1} = \begin{pmatrix} 1/3 & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & 1/3 \end{pmatrix}$$

Our aim is to show that ϕ and ρ are in fact the generators of a free subgroup of SO_3. By the definition of a free group, we must show that, after simplifying, all finite products formed from elements of $\{\phi^{\pm 1}, \rho^{\pm 1}\}$ are unique. It is equivalent to show that no such finite product equals the identity, e, except for the trivial examples $\rho\rho^{-1}$, $\rho^{-1}\rho$, $\phi^{-1}\phi$, and $\phi\phi^{-1}$. To see why, suppose the contrary. That is, suppose there exists a finite product in simplified form $\prod_{i=1}^{n} x_i = e$ such that $x_i \in \{\phi^{\pm 1}, \rho^{\pm 1}\}$ for all $1 \leq i \leq n$ where $n \geq 2$. Then $\left(\prod_{i=1}^{n-1} x_i\right) x_n = e$, so $\prod_{i=1}^{n-1} x_i = x_n^{-1}$. Hence $x_n^{-1} \in \{\phi^{\pm 1}, \rho^{\pm 1}\}$ is not unique. Because this argument can be reversed, the converse also holds. Fortunately, we need only consider those products w

ending with ϕ or ϕ^{-1} because if w does not already end with ϕ or ϕ^{-1}, we have $w \neq e$ precisely when $w\phi \neq e$. For the remainder of this proof, the products w and v are assumed to end in ϕ or ϕ^{-1}.

We will now show that for any nontrivial product of finitely many matrices $w = \prod_{i=1}^{n} x_i$, where $n \in \mathbb{N}$ and $x_i \in \{\phi^{\pm 1}, \rho^{\pm 1}\}$ for all $1 \leq i \leq n$, we have $w[\begin{array}{ccc} 1 & 0 & 0 \end{array}]^T \neq [\begin{array}{ccc} 1 & 0 & 0 \end{array}]^T$. This will demonstrate that w cannot be the identity, proving that each nontrivial w is unique. We proceed by showing that for any nontrivial w, the product $w[\begin{array}{ccc} 1 & 0 & 0 \end{array}]^T = 1/3^k[\begin{array}{ccc} a & b\sqrt{2} & c \end{array}]^T$ where a, b, c are integers and 3 does not divide c.

To show that a, b, c are integers for any w, we employ finite induction. For the inductive base, suppose w is either ϕ or ϕ^{-1}. Clearly, $w[\begin{array}{ccc} 1 & 0 & 0 \end{array}]^T = 1/3[\begin{array}{ccc} 1 & \pm 2\sqrt{2} & 0 \end{array}]^T$. Now suppose w is of length k and we have $w[\begin{array}{ccc} 1 & 0 & 0 \end{array}]^T = 1/3^k[\begin{array}{ccc} a & b\sqrt{2} & c \end{array}]^T$ where a, b, c are integers. Then the products $\phi^{\pm 1}w$ and $\rho^{\pm 1}w$ are as follows:

$$\begin{array}{rcl} \phi^{\pm 1}w & = & 1/3^{k+1}[\begin{array}{ccc} a \mp 4b & (b \pm 2a)\sqrt{2} & 3c \end{array}]^T \\ \rho^{\pm 1}w & = & 1/3^{k+1}[\begin{array}{ccc} 3a & (b \mp 2c)\sqrt{2} & c \pm 4b \end{array}]^T \end{array}$$

Clearly since, a, b, c are integers, so are $a \mp 4b, b \pm 2a, 3c, 3a, b \mp 2c, c \pm 4b$. Hence, by the principle of mathematical induction, w is always of the form $1/3^k[\begin{array}{ccc} a & b\sqrt{2} & c \end{array}]^T$ where a, b, c are integers.

Now we need only show that b is never divisible by 3. There are four cases and we will again employ an inductive argument, showing that if b is not divisible by three for a given product, left multiplication by $\phi^{\pm 1}$ or $\rho^{\pm 1}$ will not change this fact. In each of the following, let v be any finite product of the form $v = \prod_{i=1}^{n} x_i$, where $n \in \mathbb{N} \cup \{0\}$ and $x_i \in \{\phi^{\pm}, \rho^{\pm}\}$ for all i. Note that this allows v to be the product of no elements whatsoever. By the above, v is of the form $1/3^k[\begin{array}{ccc} a & b\sqrt{2} & c \end{array}]^T$ a, b, c are integers.

(i) Suppose $w = \phi^{\pm 1}\rho^{\pm 1}v$. From the above, we have $\rho^{\pm 1}v = 1/3^k[\begin{array}{ccc} 3a & (b \mp 2c)\sqrt{2} & c \pm 4b \end{array}]^T$. Now let $a' = 3a$, $b' = b \mp 2c$, and $c' = c \pm 4b$ so that $\rho^{\pm 1}v = 1/3^{k+1}[\begin{array}{ccc} a' & b'\sqrt{2} & c' \end{array}]^T$. Multiplying by $\phi^{\pm 1}$ yields $w = \phi^{\pm 1}\rho^{\pm 1}v = 1/3^{k+1}[\begin{array}{ccc} a' \mp 4b' & (b' \pm 2a')\sqrt{2} & 3c' \end{array}]^T$. Now let $b'' = b' \pm 2a'$. Since $a' = 3a$ and $b'' = b' \pm 2a'$, $3|b''$ precisely when $3|b'$.

(ii) Suppose $w = \rho^{\pm 1}\phi^{\pm 1}v$. From the above, we have $\phi^{\pm 1}v = 1/3^k[\begin{array}{ccc} a \mp 4b & (b \pm 2a)\sqrt{2} & 3c \end{array}]^T$.

Now let $a' = a \mp 4b$, $b' = b \pm 2a$, and $c' = 3c$ so that $\phi^{\pm 1}v = 1/3^{k+1}[\ a' \quad b'\sqrt{2} \quad c'\ ]^T$. Multiplying by $\rho^{\pm 1}$ yields $w = \rho^{\pm 1}\phi^{\pm 1}v = 1/3^{k+1}[\ 3a' \quad (b' \mp 2c')\sqrt{2} \quad c' \pm 4b'\ ]^T$. Now let $b'' = b' \mp 2c'$. Since $c' = 3c$ and $b'' = b' \mp 2c'$, $3|b''$ precisely when $3|b'$.

(iii) Suppose $w = \phi^{\pm 1}\phi^{\pm 1}v$. From the above, we have $\phi^{\pm 1}v = 1/3^k[\ a \mp 4b \quad (b \pm 2a)\sqrt{2} \quad 3c\ ]^T$. Now let $a' = a \mp 4b$, $b' = b \pm 2a$, and $c' = 3c$ so that $\phi^{\pm 1}v = 1/3^{k+1}[\ a' \quad b'\sqrt{2} \quad c'\ ]^T$. Multiplying by $\phi^{\pm 1}$ yields $w = \phi^{\pm 1}\phi^{\pm 1}v = 1/3^{k+1}[\ a' \mp 4b' \quad (b' \pm 2a')\sqrt{2} \quad 3c'\ ]^T$. Now let $b'' = b' \pm 2a'$. Substituting and rearranging,

$$\begin{aligned} b'' &= b' \pm 2a' \\ &= b' \pm 2(a \mp 4b) \\ &= b' \pm 2a - 8b \\ &= b' + (b \pm 2a) - 9b \end{aligned}$$

Since $b \pm 2a = b'$, we have $b'' = 2b' - 9b$. Hence, $3|b''$ precisely when $3|b'$.

(iv) Suppose $w = \rho^{\pm 1}\rho^{\pm 1}v$. From the above, we have $\rho^{\pm 1}v = 1/3^k[\ 3a \quad (b \mp 2c)\sqrt{2} \quad c \pm 4b\ ]^T$. Now let $a' = 3a$, $b' = b \mp 2c$, and $c' = c \pm 4b$ so that $\rho^{\pm 1}v = 1/3^{k+1}[\ a' \quad b'\sqrt{2} \quad c'\ ]^T$. Multiplying by $\rho^{\pm 1}$ yields $w = \rho^{\pm 1}\rho^{\pm 1}v = 1/3^{k+1}[\ 3a' \quad (b' \mp 2c')\sqrt{2} \quad c' \pm 4b'\ ]^T$. Now let $b'' = b' \mp 2c'$. Substituting and rearranging,

$$\begin{aligned} b'' &= b' \mp 2c' \\ &= b' \mp 2(c \pm 4b) \\ &= b' \mp 2c - 8b \\ &= b' + (b \mp 2c) - 9b \end{aligned}$$

Since $b \mp 2c = b'$, we have $b'' = 2b' - 9b$. Hence, $3|b''$ precisely when $3|b'$.

Hence, in each of the four cases, we have $3|b''$ precisely when $3|b'$. For the inductive base, recall that $w[\ 1 \quad 0 \quad 0\ ]^T = 1/3[\ 1 \quad \pm 2\sqrt{2} \quad 0\ ]^T$. Therefore, by the principle of mathematical induction, b is never divisible by 3 so our desired result follows. $\square$

Theorem 3.2 (The Hausdorff Paradox)**.** *There exists a countable subset, D, of the unit sphere, S^2, such that $S^2 \backslash D$ is paradoxical with respect to SO_3.*

Proof. Our strategy is to combine Corollary 2.1 with Theorem 3.1. However, to do so we must first consider all points of S^2 that remain nontrivially fixed under the action of the free group, call it F, constructed in Theorem 3.1. Recall that the elements of F are rotation matrices. Hence we must consider all the points on the unit sphere that are fixed under the nonidentity rotations in F.

The question we must ask ourselves is which points remain fixed when the unit sphere is (nontrivially) rotated about an arbitrary axis? Geometrically it is clear that when a sphere is rotated, exactly two points remain fixed: the points where the axis of rotation intersects the sphere. Thus, ignoring the identity rotation, each element of F fixes exactly two points in S^2. Let D be the collection of points that are fixed under some nonidentity rotation in F. F is countable because we can enumerate its elements as follows:

$$\begin{array}{rl}
e & \\
\rho, \rho\rho, \rho\phi, \rho\phi^{-1}, \rho\rho\rho, \rho\rho\phi, \rho\rho\phi^{-1} & \dots \\
\rho^{-1}, \rho^{-1}\rho^{-1}, \rho^{-1}\phi, \rho^{-1}\phi^{-1}, \rho^{-1}\rho^{-1}\rho^{-1}, \rho^{-1}\rho^{-1}\phi, \rho^{-1}\rho^{-1}\phi^{-1} & \dots \\
\phi, \phi\phi, \phi\rho, \phi\rho^{-1}, \phi\phi\phi, \phi\phi\rho, \phi\phi\rho^{-1}, & \dots \\
\phi^{-1}, \phi^{-1}\phi^{-1}, \phi^{-1}\rho, \phi^{-1}\rho^{-1}, \phi^{-1}\phi^{-1}\phi^{-1}, \phi^{-1}\phi^{-1}\rho, \phi^{-1}\phi^{-1}\rho^{-1}, & \dots
\end{array}$$

Hence, since D has exactly two elements corresponding to each element of F, D is also countable. Now let $p \in S^2 \backslash D$ and $g \in F \backslash \{e\}$. We want to show that $gp \in S^2 \backslash D$. That is, we need to show that $S^2 \backslash D$ is closed under the action of F. Suppose the contrary. Then for some $h \in F$, $h(gp) \in D$ which means that $h(gp) = gp$. But if this is the case, then $g^{-1}hg(p) = p$. Since F is a group and $g, h \in F$, $g^{-1}hg \in F$. This implies that p is not a member of $S^2 \backslash D$, which is a contradiction. Thus $S^2 \backslash D$ is closed under the action of F.

We can now apply Corollary 2.1 and Theorem 3.1 to $S^2 \backslash D$. Since F is a free group of order 2 acting on $S^2 \backslash D$ with only trivial fixed points, $S^2 \backslash D$ is F-paradoxical. Since F is by construction a subgroup of SO_3, the definition of a paradoxical decomposition (Definition 2.1) implies that $S^2 \backslash D$ is also SO_3 paradoxical. ☐

We would now like to explore the measure-theoretic consequences of the Hausdorff paradox as we did with the paradox on the unit circle described at the end of Section 2. We begin with a definition.

Definition 3.1. *Suppose a group G acts on a set X and $E \subseteq X$. Then E is called G-negligible if and only if $\mu(E) = 0$ for any G-invariant measure μ on $\mathcal{P}(X)$ such that $\mu(E) < \infty$.*

Theorem 3.3. *If a set E is paradoxical with respect to a group G, it is G-negligible.*

Proof. Suppose that E is G-paradoxical. Then we have $E = \bigcup_i^m g_i A_i = \bigcup_j^n h_j B_j$ for some pairwise disjoint collection $A_1, \ldots, A_m, B_1, \ldots, B_n \subseteq E$ and some elements $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$. Now suppose that μ is a G-invariant measure on $\mathcal{P}(X)$ such that $\mu(E) < \infty$. Since $A_1, \ldots, A_m, B_1, \ldots, B_n$ are pairwise disjoint, $\{\bigcup_i^m A_i\} \bigcup \{\bigcup_j^n B_j\}$ is at most E. Hence, $\mu(E) \geq \mu(\{\bigcup_i^m A_i\} \bigcup \{\bigcup_j^n B_j\})$. Invoking σ-additivity, we have $\mu(E) \geq \sum_{i=1}^m \mu(A_i) + \sum_{j=1}^n \mu(B_j)$ Because we have assumed that μ is G-invariant, $\sum_{i=1}^m \mu(A_i) + \sum_{j=1}^n \mu(B_j) = \sum_{i=1}^m \mu(g_i A_i) + \sum_{j=1}^n \mu(h_j B_j)$. Further, by σ-subadditivity and the fact that E is G-paradoxical, we have:

$$\begin{aligned}
\sum_{i=1}^m \mu(g_i A_i) &\geq \mu\left(\bigcup_{i=1}^m g_i A_i\right) = \mu(E) \\
\sum_{j=1}^n \mu(h_j B_j) &\geq \mu\left(\bigcup_{j=1}^n h_j B_j\right) = \mu(E)
\end{aligned}$$

Combining the above, $\mu(E) \geq \mu(E) + \mu(E) = 2\mu(E)$, which is only possible if $\mu(E) = \infty$ or $\mu(E) = 0$. By assumption $\mu(E) < \infty$, therefore $\mu(E) = 0$. □

One might naturally wonder whether the converse of the above statement holds. In particular, if a given set E is not G-paradoxical, does a G-invariant measure on $\mathcal{P}(X)$ exist such that $\mu(E) < \infty$? The answer turns out to be yes, though a proof of this result, Tarski's Theorem, is beyond the scope of the present discussion. For details see Wagon (1993). We are now prepared to say something about the existence of finite measures as a consequence of the Hausdorff paradox.

Theorem 3.4. *The unit sphere is SO_3-negligible. Thus, there is no finite, rotation-invariant measure on the unit sphere S^2.*

Proof. Suppose that μ is a finite, rotation invariant measure on $\mathcal{P}(S^2)$ and let D be defined as in Theorem 3.2. Since we have shown that $S^2 \backslash D$ is SO_3-paradoxical (Theorem 3.2), Theorem 3.3 establishes that $\mu(S^2 \backslash D) = 0$. Since μ is a measure, by σ-additivity $\mu(S^2) = \mu(S^2 \backslash D) + \mu(D)$. Hence, since $\mu(S^2 \backslash D) = 0$, we have $\mu(S^2) = \mu(D)$. Thus we need only show that $\mu(D) = 0$ to prove that S^2 is SO_3 negligible.

To being, consider a line λ passing through the origin that does not intersect any points of D. We know that such a λ exists because D is countable while the set of all lines passing through the origin is uncountable. For any $p \in D$, define by $A(p)$ the set of all unique angles θ (i.e. taking θ and $\theta + 2\pi$ as the same angle) such that when p is rotated θ radians about the line λ, p is mapped to another element of D. Since D is countable, so is $A(p)$ for any $p \in D$. Thus, because the union of countably many countable sets is itself countable, $A = \cup\{A(p) : p \in D\}$ is countable. Since there are uncountably many rotations about the line λ, we can choose an angle ρ that is not an element of A. Thus, by definition, rotating a point $d \in D$ by ρ radians about λ maps d to a point not in D. More formally, $\rho(D) \cap D = \emptyset$.

Since $\rho(D) \cup D \subseteq S^2$, the monotonicity of μ implies that $\mu(S^2) \leq \mu(D \cup \rho(D))$. Because D and $\rho(D)$ are disjoint, we may further apply σ-additivity, yielding $\mu(S^2) \leq \mu(D) + \mu(\rho(D))$. By SO_3-invariance we have that $\mu(\rho(D)) = \mu(D)$. Hence, $\mu(S^2)/2 \geq \mu(D)$.

We may now repeat exactly the same argument, subsituting $D \cup \rho(D)$ for D as follows. First choose a ρ' such that $(D \cup \rho(D)) \cap \rho'(D \cup \rho(D)) = \emptyset$. This is accomplished by choosing any ρ' that is not an element of A such that $\rho' \neq \rho$. Following the same steps as above yields $\mu(S^2)/4 \geq \mu(D)$. This process can be repeated indefinitely, taking $D \cup \rho(D) \cup \rho'(D)$ for D, $D \cup \rho(D) \cup \rho'(D) \cup \rho''(D)$ for D, etc. Hence, for any $k \in \mathbb{N}$, $\mu(D) \leq \mu(S^2)/2^k$. Therefore, we must have $\mu(D) = 0$. $\square$

The Hausdorff paradox tells us all that we need to know about finite, rotation-invariant measures on the unit sphere. Indeed, it shows that such measures do not exist. However there is a related, and somewhat more striking result that extends Hausdorff's paradoxical decomposition to the entire sphere. This is the famous Banach-Tarski paradox. Intuitively, the theorem states that it is possible to cut a ball into finitely many pieces, subject those pieces to rigid (i.e. volume preserving) motions, and reassemble them into *two* balls, each the size of the first! Because the proof requires considerably more machinery than we have developed here, in particular the notion of equi-decomposability, and does not materially add to our understanding of measure theory, we pass over the details. For a humorous, intuitive introduction see Blumenthal (1940). For a formal proof, see Wagon (1993).

We have shown that there is no translation-invariant, finite measure defined on all subsets of the unit interval, and no rotation-invariant finite measure defined on all subsets of the unit circle or the unit sphere. All of these results come as a direct consequence of the Axiom of Choice, which allows us to define particularly nasty sets without specifying them directly. Useful though it is, measure theory is inherently limited.

References

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