Continued Fractions and Ergodic Theory

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1 Introduction

This paper provides an introduction to the fascinating subject of continued fractions. Although continued fractions arise in a number of different guises throughout pure and applied mathematics, we concentrate here on the regular continued fractions with natural elements, a particularly elegant way of representing the real numbers. The formal prerequisites for this paper are slight; we assume only that the reader is familiar with the basic ideas of measure theory, introducing ancillary results and definitions as needed. The remainder of the paper is organized as follows. Section two develops the basic theory of continued fractions through a series of examples and proofs of the most important results. Section three introduces some ideas from ergodic theory and uses them to investigate a dynamical system arising from the continued fraction representation of the real numbers. Section four concludes.

2 Thirteen Ways of Looking at a Number

Numbers are represented in many ways; VIII, 8, and 1000 refer to the same concept in the Roman numeral, decimal, and binary systems, respectively. The choice between them is merely a question of application. Roman numerals are fine for numbering a clock face, but impractical when balancing a checkbook. The decimal system is convenient for people, but not for computers. In a similar vein, continued fractions provide an alternative to the familiar decimal expansion as a way to represent arbitrary real numbers. As we shall see, this representation has some very convenient properties.

2.1 Some Preliminaries

Definition 2.1.1 (Regular Continued Fraction)**.** *Let $x \in \mathbb{R}$. The sequence $\{a_n\}_{n\in\mathbb{N}_0}$ such that*

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ \dots}}$$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$. This is called the regular continued fraction representation of x, written in more compact notation as $x = [a_0; a_1, a_2, \ldots]$.

At this point, the above definition likely raises more questions than it answers. For example, do all real numbers have a continued fraction representation? Is there an algorithm for producing continued fraction representations for arbitrary real numbers? Is this representation unique? The answer to all three turns out to be yes. However, before delving into general principles, we begin with a simple example to illustrate the connection between continued fractions and decimal expansions. Consider the following continued fraction:

$$[2; 1, 2, 2] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}$$

A little arithmetic reveals:

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}} = 2 + \cfrac{1}{1 + \cfrac{2}{5}} = 2 + \frac{5}{7} = \frac{19}{7} \approx 2.71428571429$$

Clearly the continued fraction given above provides a representation of the rational number given by the decimal expansion 2.71428571429, namely 19/7. Then again, so does the following:

$$[2; 1, 2, 1, 1] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

In order for continued fraction representations to be unique, we require that if $\{a_n\}$ has finitely many terms, the last one may not equal one.

Definition 2.1.2 (Remainder)**.** *For a continued fraction* $x = [a_0, a_1, a_2, \ldots]$*, the quantity*

$$r_k = [a_k; a_{k+1}, a_{k+2}, \ldots]$$

for $k \in \mathbb{N}$ *is called the* k*th order remainder of* x*.*

Returning to the example given above, the third order remainder of $[2; 1, 2, 2]$ is given by

$$[2; 2] = 2 + \frac{1}{2}$$

By definition, $a_k \in \mathbb{N}$ for $k \in \mathbb{N}$, and as mentioned above we require that the last term of $\{a_n\}$ must not equal one when the sequence has finitely many terms. Thus, the kth order remainder of x is strictly greater than one, and may be arbitrarily large.

Definition 2.1.3 (Convergents)**.** *For a continued fraction* $x = [a_0, a_1, a_2, \ldots]$*, the quantity*

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$$

where $p_n/q_n \in \mathbb{Q}$ *is called the* n*th order convergent (or approximant) of* x*.*

Again returning to the above example, the second order convergent of $[2; 1, 2, 2]$ is given by:

$$\frac{p_3}{q_3} = [2; 1, 2] = 2 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{8}{3} \approx 2.6667$$

To summarize, the nth order convergent of x gives all the terms of the continued fraction expansion of x up to and including a_n, while the kth order remainder of x give all the terms from a_k onwards. We are now ready to introduce the continued fraction algorithm, which gives unique representations of arbitrary real numbers. In the interest of space, we omit the proof of uniqueness. For details, see Khinchin [2].

Theorem 2.1.1. *For any* $x \in \mathbb{R}$ *there exists a unique continued fraction representation* $[a_0; a_1, a_2, \cdots] = x$*. If* x *is rational this fraction has finitely many terms, otherwise it has infinitely many terms.*

The continued fraction algorithm proceeds as follows. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} a_0 &= \lfloor x \rfloor \\ r_1 &= 1/(x - a_0) \\ a_1 &= \lfloor r_1 \rfloor \\ r_2 &= 1/(r_1 - a_1) \\ a_2 &= \lfloor r_2 \rfloor \\ &\vdots \\ r_n &= 1/(r_{n-1} - a_{n-1}) \\ a_n &= \lfloor r_n \rfloor \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function (i.e. $\lfloor x \rfloor$ gives the greatest integer $n \leq x$). Again, when a continued fraction has finitely many terms, the last term may not be one. This process is best illustrated with an example. Suppose $x = 18/5$. Then,

$$x = 3 + \frac{3}{5} = 3 + \frac{1}{5/3} = 3 + \frac{1}{1 + 2/3} = 3 + \frac{1}{1 + \frac{1}{3/2}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}$$

Hence, $18/5 = [3; 1, 1, 2]$. We can see quite easily that this process will end after finitely many steps provided that x is rational. However, if x is irrational it will continue forever.

2.2 A Computational Example: $\sqrt{2}$

Finding the continued fraction representation of a rational number is easy enough; the process involves finitely many steps of simple arithmetic. However it may not be clear how to proceed in the case of the irrationals. Fortunately, it is sometimes possible to construct a simple recursive argument. To see how this works, we take a simple example: $\sqrt{2}$. After noting that $1 < \sqrt{2} < 3$ (i.e. $a_0 = 1$), the rest is straightforward algebra. The second step of the expansion is as follows:

$$
\begin{aligned}
\sqrt{2} &= 1+(\sqrt{2}-1) = 1+\cfrac{1}{\cfrac{1}{\sqrt{2}-1}} \\
&= 1+\cfrac{1}{\left(\cfrac{1}{\sqrt{2}-1}\right)\left(\cfrac{\sqrt{2}+1}{\sqrt{2}+1}\right)} \\
&= 1+\frac{1}{\sqrt{2}+1} \\
&= 1+\frac{1}{2+(\sqrt{2}-1)}
\end{aligned}
$$

But now we are left to expand exactly what we started with: $(\sqrt{2}-1)$. Hence, proceeding recursively:

$$
\sqrt{2} = 1+\cfrac{1}{2+\cfrac{1}{2+\ \ldots}}
$$

or using more compact notation, $\sqrt{2} = [1; 2, 2, \ldots]$. To get a sense of what this expansion means, we can compute the first few convergents as follows:

$$
\begin{aligned}
\frac{p_1}{q_1} &= [1;2] = \frac{3}{2} = 1.5 \\
\frac{p_2}{q_2} &= [1;2,2] = \frac{7}{5} = 1.4 \\
\frac{p_3}{q_3} &= [1;2,2,2] = \frac{17}{12} \approx 1.41667 \\
\frac{p_4}{q_4} &= [1;2,2,2,2] = \frac{41}{29} \approx 1.41379 \\
\frac{p_5}{q_5} &= [1;2,2,2,2,2] = \frac{99}{70} \approx 1.41429
\end{aligned}
$$

Intuitively, the convergents can be thought of as a sequence of "best" rational approximations to $\sqrt{2}$ given certain limitations on the size of the numerator and denominator. As we can see, the convergents appear to be alternating about the true value of approximately 1.41421356237. Before we can say anything more precise, however, we need to derive a few formal results.

2.3 A Little Theory

Theorem 2.3.1. *For any $n \in \mathbb{N}_0$, the following hold:*

$$p_{n+1} = a_{n+1}p_n + p_{n-1} \tag{1}$$

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \tag{2}$$

where $p_{-1} = q_0 = 1, q_{-1} = 0$, and $p_0 = a_0$.

Proof. We proceed by induction. First, we examine the case where $n = 0$. Clearly,

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_1a_0 + 1}{a_1} = \frac{a_1p_0 + p_{-1}}{a_1q_0 + q_{-1}}.$$

Now suppose the result holds for all $k < n$. Now consider the following:

$$\frac{p'_{n-1}}{q'_{n-1}} = [a_1; a_2, \ldots, a_n]$$

$$\frac{p'_{n-2}}{q'_{n-2}} = [a_1; a_2, \ldots, a_{n-1}]$$

$$\frac{p'_{n-3}}{q'_{n-3}} = [a_1; a_2, \ldots, a_{n-2}]$$

Since each of these is a convergent of order less than n, the inductive hypothesis applies to all three. Thus, $p'_{n-1} = a_{n-1}p'_{n-2} + p'_{n-3}$. However, the $(n-1)$th element of $[a_1; a_2, \ldots, a_n]$ is in fact a_n because our convention for expressing continued fractions includes a zeroth element. Extending the same argument to q'_{n-1},

$$\frac{p'_{n-1}}{q'_{n-1}} = \frac{a_np'_{n-2} + p'_{n-3}}{a_nq'_{n-2} + q'_{n-3}} \tag{3}$$

We know by definition that

$$\frac{p_{n-1}}{q_{n-1}} = a_0 + \frac{1}{[a_1; a_2, \ldots, a_{n-1}]} = a_0 + \frac{q'_{n-2}}{p'_{n-2}},$$

hence

$$\frac{p'_{n-1}}{q'_{n-1}} = \frac{a_0 p'_{n-2} + q'_{n-2}}{p'_{n-2}}. \tag{4}$$

A similar argument shows that

$$\frac{p'_{n-2}}{q'_{n-2}} = \frac{a_0 p'_{n-3} + q'_{n-3}}{p'_{n-3}}. \tag{5}$$

Clearly,

$$\frac{p'_n}{q'_n} = a_0 + \frac{1}{[a_1; a_2, \ldots, a_{n-1}]} = a_0 + \frac{q'_{n-1}}{p'_{n-1}}.$$

Applying (3) and rearranging,

$$\begin{aligned}
\frac{p'_n}{q'_n} &= a_0 + \frac{q'_{n-1}}{p'_{n-1}} \\
&= a_0 + \frac{a_n q'_{n-2} + q'_{n-3}}{a_n p'_{n-2} + p'_{n-3}} \\
&= \frac{a_0(a_n p'_{n-2} + p'_{n-3}) + (a_n q'_{n-2} + q'_{n-3})}{a_n p'_{n-2} + p'_{n-3}} \\
&= \frac{(a_0 p'_{n-3} + q'_{n-3}) + a_n(a_0 p'_{n-2} q'_{n-2})}{a_n p'_{n-2} + p'_{n-3}}
\end{aligned}$$

But by (4), we have $p_{n-1} = a_0 p'_{n-2} q'_{n-2}$ and $q_{n-1} = p'_{n-2}$. Further, by (5) we have $p_{n-2} = a_0 p'_{n-3} + q'_{n-3}$ and $q_{n-2} = p'_{n-3}$. Therefore,

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$

□

Theorem 2.3.2. *For any $n \in \mathbb{N}_0$,*

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n,$$

where $p_{-1} = q_0 = 1, q_{-1} = 0,$ and $p_0 = a_0$.

Proof. Multiply (1) by q_n and (2) by p_n, yielding

$$q_n p_{n+1} = q_n(a_{n+1} p_n + p_{n-1}) \tag{6}$$

$$p_n q_{n+1} = p_n(a_{n+1}q_n + q_{n-1}), \tag{7}$$

Now subtract (6) from (7). We have

$$\begin{aligned} q_{n+1}p_n - p_{n+1}q_n &= p_n(a_{n+1}q_n + q_{n-1}) - q_n(a_{n+1}p_n + p_{n-1}) \\ &= p_n q_{n-1} - q_n p_{n-1} \\ &= -(q_n p_{n-1} - p_n q_{n-1}) \end{aligned}$$

Now apply this formula to itself, yielding

$$q_{n+1}p_n - p_{n+1}q_n = -(q_n p_{n-1} - p_n q_{n-1}) = (q_{n-1}p_{n-2} - p_{n-2}q_{n-2})$$

If we repeat this process $n+1$ times, we will eventually reach $q_0 p_{-1} - p_0 q_{-1}$ which is simply $1 - 0 = 1$. At each step, the expression is multiplied by an additional -1. Therefore,

$$q_{n+1}p_n - p_{n+1}q_n = (-1)^n$$

□

Corollary 2.3.1. *For any $n \in \mathbb{N}_0$,*

$$\frac{q_n p_{n-1}}{q_{n-1}} - \frac{p_n q_{n-1}}{q_n} = \frac{(-1)^n}{q_n q_{n-1}},$$

where $p_{-1} = q_0 = 1, q_{-1} = 0,$ and $p_0 = a_0$.

Proof. Simply divide both sides of the equality given in the previous theorem by $q_n q_{n-1}$. The result follows. □

Theorem 2.3.3. *For any $n \in \mathbb{N}_0$,*

$$q_{n+1}p_{n-1} - p_{n+1}q_{n-1} = a_{n+1}(-1)^n,$$

where $p_{-1} = q_0 = 1, q_{-1} = 0,$ and $p_0 = a_0$.

Proof. Multiply (1) by q_{n-1} and (2) by p_{n-1}, yielding

$$q_{n-1}p_{n+1} = q_{n-1}(a_{n+1}p_n + p_{n-1}) \tag{8}$$

$$p_{n-1}q_{n+1} = p_{n-1}(a_{n+1}q_n + q_{n-1}). \tag{9}$$

Now subtract (8) from (9):

$$\begin{aligned} q_{n+1} - p_{n-1} - p_{n+1}q_{n-1} &= p_{n-1}(a_{n+1}q_n + q_{n-1}) - q_{n-1}(a_{n+1}p_n + p_{n-1}) \\ &= a_{n+1}(q_n p_{n-1} - p_n q_{n-1}) \end{aligned}$$

By Theorem 2.3.2, $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$. Substituting:

$$q_{n+1}p_{n-1} - q_{n-1}p_{n+1} = a_{n+1}(-1)^n$$

□

Corollary 2.3.2. *For any $n \in \mathbb{N}_0$,*

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}(-1)^n}{q_{n+1}q_{n-1}},$$

where $p_{-1} = q_0 = 1, q_{-1} = 0$, and $p_0 = a_0$.

Proof. Simply divide both sides of the equality given in the previous theorem by $q_{n+1}q_{n-1}$. □

Theorem 2.3.4 (The Nature of the Convergents)**.** *The odd order convergents of a continued fraction form an increasing sequence, while the even order convergents form a decreasing sequence. Further, any odd order convergent is greater than any even order convergent.*

Proof. First note that by definition, $a_n \in \mathbb{N}$, for any $n > 0$. Therefore, for any $n > 0$ the expression given in the preceding corollary will be positive precisely when n is even, and negative precisely when n is odd. It follows that the odd order convergents form an increasing sequence, and the evens a decreasing sequence (A). Similarly, we see from Corollary 2.3.1 that any odd convergent is greater than the even convergent that immediately follows it (B).

Now consider an arbitrary odd order convergent p_{2k-1}/q_{2k-1} and an arbitrary even order convergent p_{2n}/q_{2n}. First, suppose that $2k - 1 > 2n$. Then by (B), $p_{2k-1}/q_{2k-1} > p_{2k}/q_{2k}$. Since, by (A), the even order convergents are decreasing, $p_{2k}/q_{2k} > p_{2n}/q_{2n}$. Hence, $p_{2k-1}/q_{2k-1} > p_{2n}/q_{2n}$. Alternatively, suppose that $2n > 2k - 1$. By (B), $p_{2n-1}/q_{2n-1} > p_{2n}/q_{2n}$. By (A), the odd order convergents form a decreasing sequence, thus $p_{2n-1}/q_{2n-1} \geq p_{2k-1}/q_{2k-1}$. Hence, $p_{2k-1}/q_{2k-1} > p_{2n}/q_{2n}$. $\square$

Theorem 2.3.5. *For* $x = [a_0; a_1, a_2, \cdots]$, $n \in \mathbb{N}$,

$$x = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}}.$$

Proof. We proceed by induction. For $n = 0$,

$$\frac{p_0 r_1 + p_{-1}}{q_0 r_1 + q_{-1}} = \frac{a_0 r_1 + 1}{r_1} = a_0 + \frac{1}{r_1} = x,$$

by definition. Now suppose the result holds for n. Then:

$$x = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p_n \left(a_{n+1} + \frac{1}{r_{n+2}}\right) + p_{n-1}}{q_n \left(a_{n+1} + \frac{1}{r_{n+2}}\right) + q_{n-1}}$$

Expanding and simplifying,

$$x = \frac{r_{n+2}(p_n a_{n+1} + p_{n-1})}{r_{n+2}(q_n a_{n+1} + q_{n-1})}.$$

By Theorem 2.3.1, we know that $p_n a_{n+1} + p_{n-1} = p_{n+1}$ and $q_n a_{n+1} + q_{n-1} = q_{n+1}$. Therefore:

$$x = \frac{p_{n+1} r_{n+2} + p_n}{q_{n+1} r_{n+2} + q_n}.$$

$\square$

2.4 Another Example: The Golden Ratio

Continued fractions give rise to some especially elegant mathematics. There is perhaps no better example than the unexpected connection between the Fibonacci numbers and the golden ratio. Conveniently enough, this relationship follows almost immediately from the results of the previous section. The golden ratio is given by the $x \in (0, 1)$ that satisfies the following equation:

$$g = \frac{1}{x} = \frac{x}{1 - x}. \tag{10}$$

Intuitively, we can imagine a line segment of unit length being divided into two pieces, one larger than the other. These two pieces satisfy the golden ratio when the ratio of the whole to the larger piece equals the ratio of the larger piece to the smaller piece. Combining (10) with the quadratic formula, a little algebra shows that $g = \frac{1+\sqrt{5}}{2}$. Popular wisdom maintains that the golden ratio has special aesthetic properties where it appears in art and nature. Though the facts do not support this assertion [4], we shall see that g has a particularly beautiful continued fraction representation.

By the definition of g, $x < 1$. Hence, $g > 1$. Expanding g in the usual way,

$$\frac{1}{x} = 1 + \left(\frac{1}{x} - 1\right) = 1 + \left(\frac{1-x}{x}\right) = 1 + \frac{1}{\frac{x}{1-x}}.$$

Since $\frac{x}{1-x} = g$, we have shown that $g = 1 + \frac{1}{g}$. Proceeding recursively,

$$g = 1 + \frac{1}{g} = 1 + \frac{1}{\frac{1}{g}} = 1 + \frac{1}{1 + \frac{1}{1+g}} = 1 + \frac{1}{1 + \frac{1}{1+\ \ldots}}$$

Hence, the continued fraction representation of the golden ratio is given by $g = [1; 1, 1, 1, \ldots]$. Clearly, real numbers that lack a periodic decimal expansion may nonetheless have a periodic continued fraction representation.

Now that we have a continued fraction representation of g, we can find the first few convergents as follows:

$$\begin{aligned}
\frac{p_1}{q_1} &= [1; 1] = \frac{2}{1} \\
\frac{p_2}{q_2} &= [1; 1, 1] = \frac{3}{2} \\
\frac{p_3}{q_3} &= [1; 1, 1, 1] = \frac{5}{3} \\
\frac{p_4}{q_4} &= [1; 1, 1, 1, 1] = \frac{8}{5} \\
\frac{p_5}{q_5} &= [1; 1, 1, 1, 1, 1] = \frac{13}{8}
\end{aligned}$$

Recalling that $p_{-1} = q_0 = 1$ and $p_0 = a_0$, we have

p_{-1}	p_0	p_1	p_2	p_3	p_4	p_5	$\dots$
1	1	2	3	5	8	13	$\dots$
q_0	q_1	q_2	q_3	q_4	q_5	q_6	$\dots$

These numbers should look familiar, as they are in fact the first seven terms of the Fibonacci sequence. On the basis of these calculations it seems reasonable to conjecture that the convergents of g are given by the ratio of consecutive terms of the Fibonacci sequence, or more formally:

$$\frac{p_n}{q_n} = \frac{F_{n-2}}{F_{n-1}},$$

where $F_n = F_{n-1} + F_{n-2}$ and $F_1 = F_2 = 1$. It turns out that this is indeed the case. Recall from the previous section that

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1} \\ q_{n+1} &= a_{n+1}q_n + q_{n-1} \end{aligned}$$

We know that in the case of the golden ratio, $a_n = 1$ for any $n \in \mathbb{N}$. Hence,

$$\begin{aligned} p_{n+1} &= p_n + p_{n-1} \\ q_{n+1} &= q_n + q_{n-1}, \end{aligned}$$

which is precisely our desired result.

3 The Ergodic Theory of Continued Fractions

The number-theoretic results of the previous section reveal a great many intriguing facts about any continued fraction. However, if we are willing to limit our attention to *almost* any continued fraction, the tools of ergodic theory can tell us even more. For the remainder of the paper we will do precisely this, examining almost every real number in the unit interval. That is, we will consider continued fractions of the form $[0; a_1, a_2, \ldots]$. Using more compact notation, we write $[a_1, a_2, a_3, \ldots]$. We begin by introducing the basic concepts of ergodic theory.

3.1 Measure Theoretic Dynamical Systems and Ergodicity

Definition 3.1.1 (Discrete Dynamical System)**.** *Let G be a semigroup with identity element e where G consists of transformations of a space X into itself. Then (X, G) is called a discrete dynamical system if and only if there exists a map $\phi : X \times G \to X$ given by $\phi(x, g) = g(x)$ for any $x \in X$, $g \in G$, such that:*

1. *$\phi(x, e) = e(x) = x$ for any $x \in X$*

2. *$\phi(g(x), h) = h(g(x))$ for any $x \in X$; $g, h \in G$.*

Note that if $G = \langle T \rangle$, that is G is generated by a single transformation, we write $(X, \langle T \rangle)$ to denote the dynamical system. This is the kind of system with which we will concern ourselves below.

Definition 3.1.2 (Measure Theoretical Dynamical System)**.** *Let $(X, \mathcal{A}, \mu)$ be a probability measure space and let (X, G) be a discrete dynamical system. Then $(X, \mathcal{A}, \mu, G)$ is called a measure theoretic dynamical system (MTDS).*

Definition 3.1.3. *Let $\Phi = (X, \mathcal{A}, \mu, \langle T \rangle)$ be a measure theoretic dynamical system generated by a single transformation T. Then:*

1. *$A \in \mathcal{A}$ is called T-invariant $\Leftrightarrow A = T^{-1}(A)$, μ-almost everywhere*

2. *Φ is called measure-preserving $\Leftrightarrow \mu(A) = \mu(T^{-1}(A))$ for any $A \in \mathcal{A}$*

3. *Φ is called ergodic $\Leftrightarrow A \in \mathcal{A}$ is T-invariant $\Rightarrow \mu(A) = 0$ or $\mu(A) = 1$*

The first two parts of the previous definition more or less speak for themselves, but the third is somewhat more subtle. Essentially, a map is ergodic if it thoroughly scrambles the points of a space. The classic conceptual example involves a bartender mixing drinks. We can think of the bartender's swizzle stick as a mapping that carries points (droplets of gin) through a space (tonic water). If this mapping is ergodic, the bartender has mixed the drink so thoroughly that every drop, no matter how small, contains at least some gin and some tonic.

3.2 The Gauss Map

To employ the definitions given above in our study of continued fractions, we require an appropriate mapping. The Gauss map, constructed from the floor function, turns out to be a particularly convenient choice.

Definition 3.2.1 (Floor and Fractional Part)**.** *The function* $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$*, such that* $\lfloor x \rfloor$ *equals the greatest integer less than or equal to* x *is called the floor function. The function* $\langle \cdot \rangle : \mathbb{R} \to \mathbb{R}$*, such that* $\langle x \rangle = x - \lfloor x \rfloor$ *is called the fractional part of* x*.*

Definition 3.2.2 (Gauss Map)**.** *The map* $T : (0, 1) \to (0, 1)$ *given by* $Tx = \left\langle \frac{1}{x} \right\rangle$ *is called the Gauss Map.*

We now consider how this map operates on an arbitrary continued fraction $x \in (0, 1]$. Suppose that $x = [a_1, a_2, a_3, \ldots]$. Then,

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ \ldots}}$$

Taking the reciprocal,

$$\frac{1}{x} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ \ldots}}$$

Clearly the floor function of $1/x$ is simply a_1. Hence, $Tx = \langle x \rangle = [a_2, a_3, a_4, \ldots]$. The Gauss map operates by lopping of the first entry in the continued fraction expansion of x, and shifting all remaining entries one place to the left.

The Gauss map has two important properties. First, it is not invertible. If, for example $x = [1, 3, 3, 3, \ldots]$ and $y = [2, 3, 3, 3, \ldots]$, $Tx = Ty = [3, 3, 3, \ldots]$ despite the fact that $x \neq y$. In fact, the inverse image of any point $z \in (0, 1)$ under the Gauss map contains infinitely many points. Second, the Gauss map is chaotic; it is highly sensitive to initial conditions. Imagine a hypothetical system whose behavior could be characterized by the Gauss map. If we were given the initial state of the system to n terms of precision, i.e. $x_0 = [a_1, a_2, \ldots, a_n]$, then we could only predict $n - 1$ steps into the future. At each iteration, the Gauss map lops off the first term. Consequently, two points that appeared to begin at the same place to the precision of our measuring device could evolve in entirely different ways beyond the $(n-1)$th step. It may be hard to imagine that such a system could exist in reality, but physicists have discovered numerous examples, from the behavior of black holes to chaotic orbits [1].

In order to apply the tools of ergodic theory to the study of continued fractions, we need to find a measure that is preserved under the Gauss map. Fortunately, such a measure does exist, and its name will not come as a surprise.

Definition 3.2.3 (Gauss Measure)**.** *For any* Ω *in the Borel sigma-algebra on the interval* $[0,1]$*, the Gauss measure of* Ω *is given by*

$$G(\Omega) = \frac{1}{\log 2} \int_{\Omega} \frac{1}{1+x} d\lambda(x)$$

where $\lambda(\cdot)$ *is the Lebesgue measure.*

It is easily verified that G is indeed a measure. Clearly $G(\emptyset) = 0$ because the integral of any function over the empty set is zero. For any measurable A, $G(A) \geq 0$ because the function $1/(1+x)$ is strictly positive on the interval $[0,1]$. By the properties of the Lebesgue integral, G is also clearly sigma-additive. Indeed, G happens to be a probability measure since $G([0,1]) = 1$. We now show that the Gauss map does in fact preserve the Gauss measure.

Theorem 3.2.1. *The Gauss map preserves the Gauss measure.*

Proof. We need only verify the result for sets of the form $[0,t]$ were $t \in (0,1)$ because these generate generate the Borel sigma-algebra $\mathcal{B}([0,1])$. Consider the inverse of $[0,t]$ under T. We have:

$$T^{-1}([0,t]) = \{x \in (0,1) : 0 \leq Tx \leq x\} = \bigcup_{n=1}^{\infty} \left[\frac{1}{n+t}, \frac{1}{n}\right].$$

To see why this relationship holds, first note that if $x = [a_1, a_2, a_3, \ldots]$, then $x \in \left(\frac{1}{a_1+1}, \frac{1}{a_1}\right]$ by the definition of continued fractions. Now consider the inverse image of x under T. Since T lops off the first entry of x and shifts all remaining entries one place to the left, T^{-1} does the opposite. Hence,

$$T^{-1}(x) = \{[n, a_1, a_2, a_3, \ldots] : n \in \mathbb{N}\}$$

Using our observation about intervals, it is clear that T^{-1} maps a "copy" of x to each interval of the form $\left(\frac{1}{n+1}\right]$. Now consider an interval around x from 0 to t where $t = [t_1, t_2, t_3, \ldots]$. Under the action of T^{-1} the endpoints are mapped as follows for any $n \in \mathbb{N}$

$$\begin{aligned} T^{-1}(0) &= \frac{1}{n} \\ T^{-1}(t) &= \cfrac{1}{n + \cfrac{1}{t_1 + \cfrac{1}{t_2 \ldots}}} = \frac{1}{n+t} \end{aligned}$$

Hence,

$$T^{-1}[0,t] = \bigcup_{n=1}^{\infty} \left[\frac{1}{n+t}, \frac{1}{n}\right]$$

By the sigma-additivity of G and the definition of the Gauss measure,

$$\begin{aligned} G(T^{-1}[0,t]) &= \sum_{n=1}^{\infty} G\left(\left[\frac{1}{n+t}, \frac{1}{n}\right]\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{\log 2} \int_{\frac{1}{n+t}}^{\frac{1}{n}} \frac{d\lambda(x)}{1+x} \\ &= \sum_{n=1}^{\infty} \frac{1}{\log 2} \left[\log\left(1 + \frac{1}{n}\right) - \log\left(1 + \frac{1}{n+t}\right)\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{\log 2} \log\left[\frac{(n+t)(n+1)}{n(n+t+1)}\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{\log 2} \left[\log\left(1 + \frac{t}{n}\right) - \log\left(1 + \frac{t}{n+1}\right)\right] \end{aligned}$$

Reversing the integration step carried out above, and again employing the definition of the Gauss measure,

$$\begin{aligned} G(T^{-1}[0,t]) &= \sum_{n=1}^{\infty} \int_{\frac{t}{n+1}}^{\frac{t}{n}} \frac{d\lambda(x)}{1+x} \\ &= \frac{1}{\log 2}\left(\int_{\frac{t}{2}}^{t} \frac{d\lambda(x)}{1+x} + \int_{\frac{t}{3}}^{\frac{t}{2}} \frac{d\lambda(x)}{1+x} + \int_{\frac{t}{4}}^{\frac{t}{3}} \frac{d\lambda(x)}{1+x} + \dots\right) \\ &= \frac{1}{\log 2}\int_{0}^{t} \frac{d\lambda(x)}{1+x} \\ &= G([0,t]) \end{aligned}$$

□

Conveniently, the Gauss map preserves the Gauss measure. Inconveniently, its definition involves taking logarithms and integrating. To avoid this difficulty we prove the following lemma, guaranteeing that any property that holds λ-almost everywhere will also hold G-almost everywhere.

Lemma 3.2.1. *The Gauss measure is comparable to the Lebesgue measure. That is, for any measurable A, there exist constants $c_1, c_2 \in \mathbb{R}^+$ such that $c_1\lambda(A) \leq G(A) \leq c_2\lambda(A)$. We write $G(A) \asymp \lambda(A)$.*

Proof. We need only examine the case where A is a set of the form $(a,b) \in [0,1]$, where $a < b$ as these generate the Borel sigma-algebra $\mathcal{B}([0,1])$. Since $1 \geq 1/(1+x)$, and $1/2 \leq 1/(1+x)$ for any $x \in [0,1]$, we have

$$\begin{aligned} \int_a^b \frac{1}{2} d\lambda(x) &\leq \int_a^b \frac{d\lambda(x)}{1+x} \leq \int_a^b 1 d\lambda(x) \\ \frac{1}{2}\lambda(a,b) &\leq \int_a^b \frac{d\lambda(x)}{1+x} \leq \lambda(a,b) \\ \frac{1}{2\log 2}\lambda(a,b) &\leq \frac{1}{\log 2}\int_a^b \frac{d\lambda(x)}{1+x} \leq \frac{1}{\log 2} \\ \frac{1}{2\log 2}\lambda(a,b) &\leq G(a,b) \leq \frac{1}{\log 2}\lambda(a,b). \end{aligned}$$

□

3.3 Ergodicity of the Gauss Map

Equipped with a map, and a measure that preserves it, we are now ready to say something more about continued fractions. As it turns out, the combination of the Gauss measure and the Gauss map acting on the unit interval forms an ergodic MTDS. The proof given below relies heavily on number-theoretic ideas developed in Section 2.3. In the interest of clarity, we first establish a few auxiliary results.

Definition 3.3.1 (n-Cylinder)**.** *For a given* $[a_1, a_2, \ldots, a_n]$*, the interval defined by*

$$I_n[a_1, a_2, \ldots, a_n] = \{[a_1, a_2, \ldots, a_n, b_{n+1}, b_{n+2}, \ldots] : b_{n+k} \in \mathbb{N}, \forall\, i > 0\}$$

is called an n-Cylinder (or fundamental interval), denoted I_n *for short.*

Lemma 3.3.1. *The quantity* s_n*, defined by* $s_n = q_{n-1}/q_n$ *satisfies the following relationship*

$$s_n = \frac{1}{[a_n; a_{n-1}, a_{n-2}, \ldots, a_1]}$$

Proof. By definition, $r_n = a_n + 1/r_{n+1}$, and by Theorem (2.3.1) $q_{n+1} = a_{n+1}q_n + q_{n-1}$. Hence, proceeding recursively,

$$\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{1}{q_n/q_{n-1}} = a_{n+1} + \cfrac{1}{a_n + \cfrac{1}{q_{n-1}/q_{n-2}}} = \ldots = a_{n+1} + \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \ldots}}$$

which is our desired result. □

Lemma 3.3.2 (Length of an n-Cylinder)**.** *For an n-Cylinder* I_n*, we have*

$$\lambda(I_n[a_1, a_2, \ldots, a_n]) = 1/q_n^2(1 + s_n)$$

Proof. By Theorem (2.3.5), we have for $x \in \mathbb{R}$,

$$x = \frac{p_n r_{n+1} + p_{n-1}}{q_n r_{n+1} + q_{n-1}} = \frac{p_n + p_{n-1}/r_{n+1}}{q_n + q_{n-1}/r_{n+1}}$$

Recall that r_n is greater than one and may be arbitrary large, and note that for an n-Cylinder, the terms $a_1, a_2, \cdots, a_n$ are fixed while r_{n+1} is allowed to vary. Further, for an n-Cylinder I_n, the quantities $p_n, q_n, p_{n-1}, q_{n-1}$ are fixed and nonnegative. Hence we can see from the above statement that the boundaries of the interval denoted by I_n are given by $r_{n+1} \to 1$ and $r_{n+1} \to \infty$, namely $\frac{p_n+p_{n-1}}{q_n+q_{n-1}}$ and $\frac{p_n}{q_n}$ respectively. Thus,

$$\lambda(I_n) = \left| \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \left| \frac{q_n p_{n-1} - p_n q_{n-1}}{q_n^2 + q_n q_{n-1}} \right| = \left| \frac{q_n p_{n-1} - p_n q_{n-1}}{q_n^2(1 + q_{n-1}/q_n)} \right|$$

Note that by Theorem (2.3.2) the numerator of this expression is equal to $(-1)^n$, and by the definition of s_n, the denominator can be expressed as $q_n^2(1 + s_n)$. Therefore,

$$\lambda(I_n) = \left| \frac{(-1)^n}{q_n^2(1 + s_n)} \right| = \frac{1}{q_n^2(1 + s_n)}$$

□

Theorem 3.3.1. *The Gauss map is ergodic with respect to the Gauss measure.*

Proof. We will begin by showing that

$$\frac{\lambda(T^{-n}[u, v] \cap I_n)}{\lambda(I_n)} \asymp \lambda[u, v]$$

for any $u, v \in [0, 1)$, $I_n \subset [0, 1)$, where $\lambda(\cdot)$ is the Lebesgue measure. From here the ergodicity of G follows quite easily.

As stated above, the map T operates by deleting the first element of a continued fraction, and then shifting all remaining elements one space to the left. To take a simple example, $T[a_1, a_2, a_3, \ldots] = [a_2, a_3, a_4, \ldots]$. Again, T^{-1} operates in reverse, shifting all elements of a continued fraction one space to the right, and allowing the now empty first element to vary arbitrary. The inverse image under T of a point $x = [a_1, a_2, a_3, \ldots]$ in the interval $[0, 1)$ is given by $T^{-1}[a_1, a_2, a_3, \ldots] = \{[k, a_1, a_2, \ldots] : k \in \mathbb{N}\}$.

Now consider the set $T^{-n}[u, v] \cap I_n$ for some fixed $I_n = [a_1, a_2, a_3, \ldots]$ and $u, v \in [0, 1)$. Any point in this set is of the form $[a_1, a_2, \ldots, a_n, b_1, b_2, \ldots]$ where $[k_1, k_2, k_3, \ldots] \in [u, v]$. To put it another way, points belonging to $T^{-n}[u, v] \cap I_n$ differ only in their $(n + 1)$th order remainders. To find the Lebesgue measure of this set, we need only find its boundaries and subtract them. If $u = [b_1, b_2, b_3, \ldots]$, and $v = [c_1, c_2, c_3, \ldots]$, then these boundaries are clearly given by $[a_1, a_2, \ldots, b_1, b_2, \ldots]$ on the left and $[a_1, a_2, \ldots, a_n, c_1, c_2, \ldots]$ on the right. Since I_n is fixed, the quantities $p_n, q_n, p_{n-1}, q_{n-1}$ are known (and in fact indentical) for every point in $T^{-n}[u,v] \cap I_n$. Hence, by expressing the boundaries in terms of convergents and remainders according to Theorem (2.3.5), we have

$$\lambda(T^{-n}[u,v] \cap I_n) = \frac{p_n v + p_{n-1}}{q_n v + q_{n-1}} - \frac{p_n u + p_{n-1}}{q_n u + q_{n-1}}$$

Combining this result with Lemma (3.3.2) and simplifying, we find that

$$\begin{aligned} \frac{\lambda(T^{-n}[u,v] \cap I_n)}{\lambda(I_n)} &= \left(\frac{p_n v + p_{n-1}}{q_n v + q_{n-1}} - \frac{p_n u + p_{n-1}}{q_n u + q_{n-1}} \right) q_n^2 (1 + s_n) \\ &= (v - u) \left[\frac{p_n q_{n-1} - p_{n-1} q_n}{(q_n v + q_{n-1})(q_n u + q_{n-1})} \right] q_n (q_n + q_{n-1}) \end{aligned}$$

Applying Theorem (2.3.2) to the numerator of this fraction, and noting that $(v - u) = \lambda[u, v]$, we have[1]

$$\frac{\lambda(T^{-n}[u,v] \cap I_n)}{\lambda(I_n)} = \lambda[u,v] \frac{q_n(q_n + q_{n-1})}{(q_n v + q_{n-1})(q_n u + q_{n-1})}$$

Thus, if we can show that

$$\frac{q_n(q_n + q_{n-1})}{(q_n v + q_{n-1})(q_n u + q_{n-1})} \asymp 1$$

we will have established that

$$\frac{\lambda(T^{-n}[u,v] \cap I_n)}{\lambda(I_n)} \asymp \lambda[u,v]$$

Setting $u = v = 1$ on the left and $u = v = 0$ on the right yields the following inequality

$$\frac{q_n^2 + q_n q_{n-1}}{q_n^2 + 2q_n q_{n-1} + q_{n-1}^2} \leq \frac{q_n(q_n + q_{n-1})}{(q_n v + q_{n-1})(q_n u + q_{n-1})} \leq \frac{q_n^2 + q_n q_{n-1}}{q_{n-1}^2}$$

[1]Because this expression is the ratio of two measures and hence must be nonnegative, we can ignore the negative sign in Theorem (2.3.2).

Simplifying and substituting the definition of s_n,

$$\frac{q_n(q_n+q_{n-1})}{(q_n+q_{n-1})(q_n+q_{n-1})} \leq \frac{q_n(q_n+q_{n-1})}{(q_nv+q_{n-1})(q_nu+q_{n-1})} \leq \left(\frac{q_n}{q_{n-1}}\right)^2 + \left(\frac{q_n}{q_{n-1}}\right)$$

$$\frac{1}{1+s_n} \leq \frac{q_n(q_n+q_{n-1})}{(q_nv+q_{n-1})(q_nu+q_{n-1})} \leq \left(\frac{1}{s_n}\right)^2 + \left(\frac{1}{s_n}\right)$$

Note that s_n is known because we have fixed $I_n = [a_1, a_2, a_3, \ldots]$. Hence we may write

$$1k_1 \leq \frac{q_n(q_n+q_{n-1})}{(q_nv+q_{n-1})(q_nu+q_{n-1})} \leq 1k_2$$

In other words,

$$\frac{q_n(q_n+q_{n-1})}{(q_nv+q_{n-1})(q_nu+q_{n-1})} \asymp 1$$

thus

$$\frac{\lambda(T^{-n}[u,v] \cap I_n)}{\lambda(I_n)} \asymp \lambda[u,v]$$

Because intervals of the form $[u, v]$ generate the Borel sigma-algebra on $[0, 1)$,

$$\frac{\lambda(T^{-n}(A) \cap I_n)}{\lambda(I_n)} \asymp \lambda(A)$$

for any $A \in \mathcal{B}([0,1))$, and any $I_n \in [0,1)$. Because the Lebesgue and Gauss measures are comparable by Lemma (3.2.1), we have

$$\frac{G(T^{-n}(A) \cap I_n)}{G(I_n)} \asymp G(A)$$

for any $A \in \mathcal{B}([0,1))$, and any $I_n \in [0,1)$. Now suppose that $A \in \mathcal{B}([0,1))$ is a T-invariant set of positive measure, i.e. $T^{-1}(A) = A$ and $G(A) > 0$. For ergodicity to be satisfied, we need to show that $G(A) = 1$. Since the n-cylinders generate $\mathcal{B}([0,1))$, we have that

$$\frac{G(A \cap B)}{G(A)} \asymp G(B)$$

for any $A, B \in \mathcal{B}([0,1))$. Taking $B = A^c$, $G(A \cap B) = G(\emptyset) = 0$. Hence, $G(B) \asymp 0$ which holds precisely when $G(B) = 0$. Therefore $G(A) = 1$, and ergodicity is satisfied. □

3.4 Consequences of Ergodicity

We have shown in the previous section that the Gauss map is ergodic with respect to the Gauss measure, but it may not be obvious why this is an interesting property. To see why this is the case, we will use of the following famous result.

Theorem 3.4.1 (The Ergodic Theorem). *Let $\Phi = (X, \mathcal{A}, \mu, G)$ be an ergodic MTDS. Then, for any $f \in L^1(X, \mu)$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int_X f d\mu$$

for μ-almost every $x \in X$.

Intuitively, the ergodic theorem gives conditions that guarantee the equality of time and space averages. Applying this result to the Gauss map yields some surprising results. For example, name any finite sequence of integers $n_1, n_2, \ldots, n_k$. By the ergodic theorem, this sequence occurs infinitely many times in the continued fraction expansion of G-almost every $x \in [0, 1)$.

To see why, notice that the sequence $n_1, n_2, \ldots, n_k$ specifies an n-cylinder $I_k([n_1, n_2, \ldots, n_k])$. Now denote by $h(x)$ an indicator function taking the value one if $x \in I_k$, and zero otherwise. The integral of h over $[0, 1)$ is simply $G(I_k)$. This quantity is strictly greater than zero because $\lambda(I_k) > 0$. Since h is an indicator function, the expression on the left hand side of the ergodic theorem gives the the proportion of time that G-almost any point x will spend in the region I_k as x is mapped infinitely many times around the unit interval under the action of T. By the equality of time and space averages, this proportion is simply $G(I_k)$. The only way that x can spend a positive proportion of time in I_k as it is mapped infinitely many times around $[0, 1)$ is if x visits I_k infinitely often. By the definition of the Gauss map, this will only occur if the sequence $n_1, n_2, \ldots, n_k$ appears infinitely many times in the continued fraction expansion of x. Note that this result holds only G-almost everywhere. The point $[1, 1, 1, \ldots]$, for example, will never visit the interval $[3, 4, 5, 6, 7, 8]$.

We now apply the ergodic theorem in a different way to consider the average size of a typical entry in some continued fraction $x \in [0, 1)$.

Theorem 3.4.2. *The average size of some entry a_n of a continued fraction $x = [a_1, a_2, a_3, \ldots]$ is infinite for G-almost every $x \in [0, 1)$.*

Proof. For a fixed $n \in \mathbb{N}$, let A_n denote the set of all continued fractions in the interval $[0, 1)$ whose first entry is n. Further, for some $x \in [0, 1)$ where $x = [a_1, a_2, a_3, \ldots]$, let $a_n(x) = a_n$. For example, if $x = [1, 2, 3, 4, \ldots]$, $a_3(x) = 3$. Note that A_n is simply the interval $\left(\frac{1}{n+1}, \frac{1}{n}\right]$. Thus by Lemma (3.2.1),

$$G(A_n) \asymp \lambda(A_n) = \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n^2+1} \asymp \frac{1}{n^2}$$

Because $a_1(x)$ is a step function taking on the value n for any $x \in A_n$,

$$\int_0^1 a_1(x)dG(x) = \sum_{n=1}^{\infty} nG(A_n)$$

But as we have seen, $G(A_n) \asymp 1/n^2$. Hence,

$$\int_0^1 a_1(x)dG(x) = \sum_{n=1}^{\infty} nG(A_n) \asymp \sum_{n=1}^{\infty} n\frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Applying the Ergodic Theorem (3.4.1),

$$\int_0^1 a_1(x)dG(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_1(T^k(x)) = \frac{1}{n}[a_1(x) + a_2(x) + a_3(x) + \ldots]$$

for G-almost every $x \in [0, 1)$. But we have shown that

$$\int_0^1 a_1(x)dG(x) = \infty$$

Our desired result follows. □

This is a striking result. But even more striking is the surprising regularity that emerges if we consider the geometric rather than arithmetic average.

Definition 3.4.1 (Khinchin's Constant)**.** *Khinchin's constant γ is given by* *the following expression*

$$\gamma = \prod_{k=1}^{\infty}\left[1+\frac{1}{k(k+2)}\right]^{\log k/\log 2} \approx 2.6854$$

Theorem 3.4.3. *The quantity* $[a_1(x)a_2(x)\cdots a_n(n)]^{\frac{1}{n}} \to \gamma$ *as* $n \to \infty$ *for* G*-almost every* $x \in [0,1)$.

Proof. To use the Ergodic theorem, we must first show that the function $\log a_1(x) \in L^1$. To see why this is the case note that

$$\int_0^1 \log a_1(x)dG(x) = \sum_{n=1}^{\infty}\log n \cdot G(A_n) \asymp \sum_{n=1}^{\infty}\log n \cdot \frac{1}{n^2} < \infty$$

We may now proceed as in the previous theorem. By the Ergodic Theorem (3.4.1) and the fact that $\log a_1(x)$ is a step function, we have

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\log a_1(T^k(x)) = \int_0^1 \log a_1(x)dG(x) = \sum_{n=1}^{\infty}\log n \cdot G(A_n)$$

G-almost everywhere. By the definition of G,

$$\begin{aligned} G(A_n) &= \frac{1}{\log 2}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{d\lambda(x)}{1+x} \\ &= \left[\frac{1}{\log 2}\cdot \log(1+x)\right]_{\frac{1}{n+1}}^{\frac{1}{n}} \\ &= \frac{1}{\log 2}\log\left[\frac{(n+1)^2}{n(n+2)}\right] \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{n}\log[a_1(x)a_2(x)\cdots a_n(x)] &= \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\log a_1(T^k(x)) \\ &= \sum_{n=1}^{\infty}\frac{\log n}{\log 2}\cdot\log\left[\frac{(n+1)^2}{n(n+2)}\right] \\ &= \sum_{n=1}^{\infty}\frac{\log n}{\log 2}\cdot\log\left[1+\frac{1}{n(n+2)}\right] \end{aligned}$$

Therefore

$$[a_1(x)a_2(x)\cdots a_n(n)]^{\frac{1}{n}} = \prod_{n=1}^{\infty}\left[1+\frac{1}{n(n+2)}\right]^{\log n/\log 2}$$

□

4 Conclusion

As we have seen, continued fractions bring to light numerous intriguing features of the real numbers, and provide a convenient setting within which to explore the main ideas of ergodic theory. There is of course far more to be said about continued fractions. Though this paper only scratches the surface of the topic, we hope that the results presented herein have given the reader a taste of this elegant branch of mathematics.

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