Problem Set #6

Econ 103

Part I – Problems from the Textbook

Chapter 4: 19, 21, 23 (*When necessary, use R rather than the Normal tables in the front of the textbook.*)

Part II – Additional Problems

1. Suppose that X is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Graph the PDF of X.

   Solution: It's an isosceles triangle with base from (0,0) to (2,0) and height 1.

   (b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

   Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\, dx &= \int_0^1 x\, dx + \int_1^2 (2-x)\, dx = \left.\frac{x^2}{2}\right|_0^1 + \left.\left(2x - \frac{x^2}{2}\right)\right|_1^2 \\ &= 1/2 + (4-2) - (2 - 1/2) = 1 \end{aligned}$$

   (c) What is $P(0.5 < X < 1.5)$?

Solution:

$$
\begin{aligned}
P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx \\
&= \left.\frac{x^2}{2}\right|_{0.5}^{1} + \left.\left(2x - \frac{x^2}{2}\right)\right|_{1}^{1.5} \\
&= (1/2 - 1/8) + (3 - 9/8) - (2 - 1/2) \\
&= 3/8 + 15/8 - 2 + 1/2 = 18/8 - 16/8 + 4/8 \\
&= 6/8 = 3/4 = 0.75
\end{aligned}
$$

2. A random variable is said to follow a Uniform(a,b) distribution if it is equally likely to take on any value in the range $[a,b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim \text{Uniform}(a,b)$.

   (a) What is the support of X?

   Solution: $[a,b]$

   (b) Explain why the PDF of X is $f(x) = 1/(b-a)$ for $a \leq x \leq b$, zero elsewhere.

   Solution: This simply generalizes the Uniform$(0,1)$ random variable from class. To capture the idea that X is equally likely to take on any value in the range $[a,b]$, the PDF must be constant. To ensure that it integrates to 1, the denominator must be $b-a$.

   (c) Using the PDF from part (b), calculate the CDF of X.

   Solution:

$$
F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \int_{a}^{x_0} \frac{dx}{b-a} = \left.\frac{x}{b-a}\right|_{a}^{x_0} = \frac{x_0 - a}{b-a}
$$

   (d) Verify that $f(x) = F'(x)$ for the present example.

   Solution:

$$
F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} = f(x)
$$

(e) Calculate $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

(f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2+a^2+ab)$.

Solution:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_a^b \frac{x^2}{b-a} = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} = \frac{b^2+a^2+ab}{3} \end{aligned}$$

(g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

Solution:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = \frac{b^2+a^2+ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2+a^2+ab}{3} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{4b^2+4a^2+4ab-3a^2-6ab-3b^2}{12} \\ &= \frac{b^2+a^2-2ab}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

3. Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$. Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.

(a) Calculate $P(-8 \le X \le 8)$.

Solution:

$$P(-8 \le X \le 8) = P(-8/4 \le X/4 \le 8/4) = P(-2 \le Z \le 2) \approx 0.95$$

where Z is a standard normal random variable.

(b) Calculate $P(0 \le Y \le 4)$.

Solution:

$$P(0 \le Y \le 4) = P\left(\frac{0-2}{2} \le \frac{Y-2}{2} \le \frac{4-2}{2}\right) = P(-1 \le Z \le 1) \approx 0.68$$

where Z is a standard normal random variable.

(c) Calculate $P(-1 \le Y \le 6)$.

Solution:

$$\begin{aligned} P(-1 \le Y \le 6) &= P\left(\frac{-1-2}{2} \le \frac{Y-2}{2} \le \frac{6-2}{2}\right) \\ &= P(-1.5 \le Z \le 2) \\ &= \Phi(2) - \Phi(-1.5) \\ &= \texttt{pnorm(2) - pnorm(-1.5)} \\ &\approx 0.91 \end{aligned}$$

where Z is a standard normal random variable.

(d) Calculate $P(X \ge 10)$.

Solution:

$$\begin{aligned} P(X \ge 10) &= 1 - P(X \le 10) = 1 - P(X/4 \le 10/4) = 1 - P(Z \le 2.5) \\ &= 1 - \Phi(2.5) = 1 - \texttt{pnorm(2.5)} \\ &\approx 0.006 \end{aligned}$$

Note: *In the following five questions* $X_1, X_2 \sim$ *iid* $N(\mu, \sigma^2)$, $Y = (X_1 - \mu)/\sigma$, $Z = (X_2 - \mu)/\sigma$.

4. (a) What is the distribution of $X_1 + X_2$?

Solution: $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$

(b) Use R to calculate $P(X_1 + X_2 > 5)$ if $\mu = 5$ and $\sigma^2 = 50$.

Solution: In this case, $X_1 + X_2 \sim N(10, 100)$, hence

$$\begin{aligned} P(X_1 + X_2 > 5) &= 1 - P(X_1 + X_2 \leq 5) \\ &= 1 - P\left(\frac{X_1 + X_2 - 10}{10} \leq \frac{5 - 10}{10}\right) \\ &= 1 - \texttt{pnorm(-0.5)} \\ &\approx 0.6914625 \end{aligned}$$

Alternatively, we could use `1- pnorm(5, mean = 10, sd = 10)`, which gives the same result.

(c) Use R to calculate the 10th percentile of the distribution of $X_1 + X_2$.

Solution: `qnorm(p = 0.1, mean = 10, sd = 10)` gives -2.815516.

5. (a) What is the distribution of Y^2?

Solution: As the sum of squares of one standard normal RV, $Y^2 \sim \chi^2(1)$.

(b) Use R to calculate $P(Y^2 \geq 1)$.

Solution:

$$P(Y^2 \geq 1) = 1 - P(Y^2 \leq 1) = 1 - \texttt{pchisq(1, df = 1)} \approx 0.3173105$$

6. (a) What is the distribution of $Y^2 + Z^2$?

Solution: Since this is the sum of squares of two independent standard normal random variables, $Y^2 + Z^2 \sim \chi^2(2)$.

(b) Use R to calculate the 95th percentile of the distribution of $Y^2 + Z^2$.

Solution: `qchisq(p = 0.95, df = 2)` gives 5.991465

7. (a) What is the distribution of $Z/\sqrt{Y^2}$?

   Solution: Since it is the ratio of a standard normal to the square root of an independent χ^2 random variable divided by its degrees of freedom (in this case one), $Z/\sqrt{Y^2} \sim t(1)$.

   (b) What value of c satisfies $P(-c \leq Z/\sqrt{Y^2} \leq c) = 0.95$?

   Solution: By the symmetry of the t-distribution, it suffices to find the 97.5th percentile (this allocates 2.5% probability to the upper and lower tails). The command `qt(p = 0.975, df = 1)` gives 12.7062, so $c \approx 12.7$. Alternatively, we could have calculated the 2.5th percentile: `qt(p = 0.025, df = 1)` gives -12.7062.

   (c) How does the interval in part (b) compare to the corresponding interval for Z?

   Solution: Since Z is a standard normal RV, $P(-2 \leq Z \leq 2) \approx 0.95$. We see that the interval for a $t(1)$ RV is *much wider* than the corresponding interval for a standard normal. In other words, extreme outcomes are much more likely under the $t(1)$ distribution.

8. (a) What is the distribution of Y^2/Z^2?

   Solution: This is the ratio of two independent χ^2 random variables, each divided by its degrees of freedom (in this case, one). Hence $Y^2/Z^2 \sim F(1,1)$.

   (b) Use R to calculate the 95th percentile of the distribution of Y^2/Z^2.

   Solution: `qf(p = 0.95, df1 = 1, df2 = 1)` gives 161.4476