Part I – Problems from the Textbook

Chapter 4: 1, 3, 5, 7, 9, 11, 13, 15, 25, 27, 29
Chapter 5: 1, 3, 5, 9, 11, 13, 17

Part II – Additional Problems

1. Suppose \( X \) is a random variable with support \(-1, 0, 1\) where \( p(-1) = q \) and \( p(1) = p \).
   
   (a) What is \( p(0) \)?
   
   Solution: By the complement rule \( p(0) = 1 - p - q \).

   (b) Calculate the CDF, \( F(x_0) \), of \( X \).

   Solution:
   
   \[
   F(x_0) = \begin{cases} 
   0, & x_0 < -1 \\
   q, & -1 \leq x_0 < 0 \\
   1 - p, & 0 \leq x_0 < 1 \\
   1, & x_0 \geq 1 
   \end{cases}
   \]

   (c) Calculate \( E[X] \).

   Solution: \( E[X] = -1 \cdot q + 0 \cdot (1 - p - q) + p \cdot 1 = p - q \)

   (d) What relationship must hold between \( p \) and \( q \) to ensure \( E[X] = 0 \)?

   Solution: \( p = q \)

2. Fill in the missing details from class to calculate the variance of a Bernoulli Random Variable directly, that is without using the shortcut formula.
Solution:

\[ \sigma^2 = \text{Var}(X) = \sum_{x \in \{0, 1\}} (x - \mu)^2 p(x) \]
\[ = \sum_{x \in \{0, 1\}} (x - p)^2 p(x) \]
\[ = (0 - p)^2(1 - p) + (1 - p)^2 p \]
\[ = p^2(1 - p) + (1 - p)^2 p \]
\[ = p^2 - p^3 + p - 2p^2 + p^3 \]
\[ = p - p^2 \]
\[ = p(1 - p) \]

3. Prove that the Bernoulli Random Variable is a special case of the Binomial Random variable for which \( n = 1 \). (Hint: compare pmfs.)

**Solution:** The pmf for a Binomial(\( n, p \)) random variable is

\[ p(x) = \binom{n}{x} p^x (1 - p)^{n-x} \]

with support \( \{0, 1, 2, \ldots, n\} \). Setting \( n = 1 \) gives,

\[ p(x) = p(x) = \binom{1}{x} p^x (1 - p)^{1-x} \]

with support \( \{0, 1\} \). Plugging in each realization in the support, and recalling that \( 0! = 1 \), we have

\[ p(0) = \frac{1!}{0!(1 - 0)!} p^0 (1 - p)^{1-0} = 1 - p \]

and

\[ p(1) = \frac{1!}{1!(1 - 1)!} p^1 (1 - p)^0 = p \]

which is exactly how we defined the Bernoulli Random Variable.

4. Suppose that \( X \) is a random variable with support \( \{1, 2\} \) and \( Y \) is a random variable with support \( \{0, 1\} \) where \( X \) and \( Y \) have the following joint distribution:

\[ p_{XY}(1, 0) = 0.20, \quad p_{XY}(1, 1) = 0.30 \]
\[ p_{XY}(2, 0) = 0.25, \quad p_{XY}(2, 1) = 0.25 \]
(a) Express the joint distribution in a $2 \times 2$ table.

**Solution:**

<table>
<thead>
<tr>
<th></th>
<th>$X$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$Y$</td>
<td>0.20</td>
<td>0.25</td>
</tr>
<tr>
<td>1</td>
<td>0.30</td>
<td>0.25</td>
</tr>
</tbody>
</table>

(b) Using the table, calculate the marginal probability distributions of $X$ and $Y$.

**Solution:**

\[
\begin{align*}
p_X(1) &= p_{XY}(1, 0) + p_{XY}(1, 1) = 0.20 + 0.30 = 0.50 \\
p_X(2) &= p_{XY}(2, 0) + p_{XY}(2, 1) = 0.25 + 0.25 = 0.50 \\
p_Y(0) &= p_{XY}(1, 0) + p_{XY}(2, 0) = 0.20 + 0.25 = 0.45 \\
p_Y(1) &= p_{XY}(1, 1) + p_{XY}(2, 1) = 0.30 + 0.25 = 0.55
\end{align*}
\]

(c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.

**Solution:** The distribution of $Y|X = 1$ is

\[
\begin{align*}
P(Y = 0|X = 1) &= \frac{p_{XY}(1, 0)}{p_X(1)} = \frac{0.2}{0.5} = 0.4 \\
P(Y = 1|X = 1) &= \frac{p_{XY}(1, 1)}{p_X(1)} = \frac{0.3}{0.5} = 0.6
\end{align*}
\]

while the distribution of $Y|X = 2$ is

\[
\begin{align*}
P(Y = 0|X = 2) &= \frac{p_{XY}(2, 0)}{p_X(2)} = \frac{0.25}{0.5} = 0.5 \\
P(Y = 1|X = 2) &= \frac{p_{XY}(2, 1)}{p_X(2)} = \frac{0.25}{0.5} = 0.5
\end{align*}
\]

(d) Calculate $E[Y|X]$. 

**Solution:**
Solution:

\[ E[Y|X = 1] = 0 \times 0.4 + 1 \times 0.6 = 0.6 \]
\[ E[Y|X = 2] = 0 \times 0.5 + 1 \times 0.5 = 0.5 \]

Hence,

\[ E[Y|X] = \begin{cases} 
0.6 & \text{with probability 0.5} \\
0.5 & \text{with probability 0.5} 
\end{cases} \]

since \( p_X(1) = 0.5 \) and \( p_X(2) = 0.5 \).

(e) What is \( E[E[Y|X]] \)?

Solution: \( E[E[Y|X]] = 0.5 \times 0.6 + 0.5 \times 0.5 = 0.3 + 0.25 = 0.55 \). Note that this equals the expectation of \( Y \) calculated from its marginal distribution, since \( E[Y] = 0 \times 0.45 + 1 \times 0.55 \). This illustrates the so-called “Law of Iterated Expectations.”

(f) Calculate the covariance between \( X \) and \( Y \) using the shortcut formula.

Solution: First, from the marginal distributions, \( E[X] = 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5 \) and \( E[Y] = 0 \cdot 0.45 + 1 \cdot 0.55 = 0.55 \). Hence \( E[X]E[Y] = 1.5 \cdot 0.55 = 0.825 \). Second,

\[ E[XY] = (0 \cdot 1) \cdot 0.2 + (0 \cdot 2) \cdot 0.25 + (1 \cdot 1) \cdot 0.3 + (1 \cdot 2) \cdot 0.25 \\
\quad = 0.3 + 0.5 = 0.8 \]

Finally \( \text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0.8 - 0.825 = -0.025 \)

5. Let \( X \) and \( Y \) be discrete random variables and \( a, b, c, d \) be constants. Prove the following:

(a) \( \text{Cov}(a+bX, c+dY) = bd \text{Cov}(X,Y) \)

Solution: Let \( \mu_X = E[X] \) and \( \mu_Y = E[Y] \). By the linearity of expectation,

\[ E[a + bX] = a + b \mu_X \]
\[ E[c + dY] = c + d \mu_Y \]

Thus, we have

\[ (a + bx) - E[a + bX] = b(x - \mu_X) \]
\[ (c + dy) - E[c + dY] = d(y - \mu_Y) \]
Substituting these into the formula for the covariance between two discrete random variables,

\[
\text{Cov}(a + bX, c + dY) = \sum_x \sum_y [b(x - \mu_X)] [d(y - \mu_Y)] p(x, y)
\]

\[= bd \sum_x \sum_y (x - \mu_X)(y - \mu_Y)p(x, y)
\]

\[= bd \text{Cov}(X, Y)
\]

(b) \(\text{Corr}(a + bX, c + dY) = \text{Corr}(X, Y)\) provided that \(b, c\) are positive.

Solution:

\[
\text{Corr}(a + bX, c + dY) = \frac{\text{Cov}(a + bX, c + dY)}{\sqrt{\text{Var}(a + bX)\text{Var}(c + dY)}}
\]

\[= \frac{bd \text{Cov}(X, Y)}{\sqrt{b^2 \text{Var}(X)d^2 \text{Var}(Y)}}
\]

\[= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
\]

\[= \text{Corr}(X, Y)
\]

6. Fill in the missing steps from lecture to prove the shortcut formula for covariance:

\[
\text{Cov}(X, Y) = E[XY] - E[X]E[Y]
\]

Solution: By the Linearity of Expectation,

\[
\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]
\]

\[= E[XY] - \mu_X Y - \mu_Y X + \mu_X \mu_Y
\]

\[= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y
\]

\[= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y
\]

\[= E[XY] - \mu_X \mu_Y
\]

\[= E[XY] - E[X]E[Y]
\]
7. Let $X_1$ be a random variable denoting the returns of stock 1, and $X_2$ be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma^2_1 = Var(X_1)$, $\sigma^2_2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A portfolio, $\Pi$, is a linear combination of $X_1$ and $X_2$ with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1 - \omega) X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0, 1]$, so that negative weights are not allowed. (This rules out short-selling.)

(a) Calculate $E[\Pi(\omega)]$ in terms of $\omega$, $\mu_1$ and $\mu_2$.

**Solution:**

\[
E[\Pi(\omega)] = E[\omega X_1 + (1 - \omega) X_2] = \omega E[X_1] + (1 - \omega) E[X_2] = \omega \mu_1 + (1 - \omega) \mu_2
\]

(b) If $\omega \in [0, 1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ and $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.

**Solution:** No. If short-selling is disallowed, the portfolio expected return must be between $\mu_1$ and $\mu_2$.

(c) Express $Cov(X_1, X_2)$ in terms of $\rho$ and $\sigma_1$, $\sigma_2$.

**Solution:**

\[
Cov(X, Y) = \rho \sigma_1 \sigma_2
\]

(d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of $\rho$, $\sigma^2_1$ and $\sigma^2_2$.)

**Solution:**

\[
Var[\Pi(\omega)] = Var[\omega X_1 + (1 - \omega) X_2] = \omega^2 Var(X_1) + (1 - \omega)^2 Var(X_2) + 2\omega(1 - \omega) Cov(X_1, X_2)
\]

\[
= \omega^2 \sigma^2_1 + (1 - \omega)^2 \sigma^2_2 + 2\omega(1 - \omega) \rho \sigma_1 \sigma_2
\]

(e) Using part (d) show that the value of $\omega$ that minimizes $Var[\Pi(\omega)]$ is

\[
\omega^* = \frac{\sigma^2_2 - \rho \sigma_1 \sigma_2}{\sigma^2_1 + \sigma^2_2 - 2 \rho \sigma_1 \sigma_2}
\]

In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio*. 

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Solution: The First Order Condition is:

\[ 2\omega\sigma_1^2 - 2(1 - \omega)\sigma_2^2 + (2 - 4\omega)\rho\sigma_1\sigma_2 = 0 \]

Dividing both sides by two and rearranging:

\[ \omega\sigma_1^2 - (1 - \omega)\sigma_2^2 + (1 - 2\omega)\rho\sigma_1\sigma_2 = 0 \]
\[ \omega\sigma_1^2 - \sigma_2^2 + \omega\sigma_2^2 + \rho\sigma_1\sigma_2 - 2\omega\rho\sigma_1\sigma_2 = 0 \]
\[ \omega(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) = \sigma_2^2 - \rho\sigma_1\sigma_2 \]

So we have

\[ \omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \]

(f) If you want a challenge, check the second order condition from part (e).

Solution: The second derivative is

\[ 2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2 \]

and, since \( \rho = 1 \) is the largest possible value for \( \rho \),

\[ 2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2 \geq 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \geq 0 \]

so the second derivative is positive, indicating a minimum. This is a global minimum since the problem is quadratic in \( \omega \).