Problem Set #5

Econ 103

Part I – Problems from the Textbook

Chapter 4: 1, 3, 5, 7, 9, 11, 13, 15, 25, 27, 29
Chapter 5: 1, 3, 5, 9, 11, 13, 17

Part II – Additional Problems

1. Suppose X is a random variable with support $\{-1, 0, 1\}$ where $p(-1) = q$ and $p(1) = p$.
   (a) What is $p(0)$?

   Solution: By the complement rule $p(0) = 1 - p - q$.

   (b) Calculate the CDF, $F(x_0)$, of X.

   Solution:

   $$F(x_0) = \begin{cases} 0, \ x_0 < -1 \\ q, \ -1 \leq x_0 < 0 \\ 1-p, \ 0 \leq x_0 < 1 \\ 1, \ x_0 \geq 1 \end{cases}$$

   (c) Calculate $E[X]$.

   Solution: $E[X] = -1 \cdot q + 0 \cdot (1 - p - q) + p \cdot 1 = p - q$

   (d) What relationship must hold between p and q to ensure $E[X] = 0$?

   Solution: $p = q$

2. Fill in the missing details from class to calculate the variance of a Bernoulli Random Variable *directly*, that is *without* using the shortcut formula.

Solution:

$$\begin{aligned}
\sigma^2 &= Var(X) = \sum_{x\in\{0,1\}} (x-\mu)^2 p(x) \\
&= \sum_{x\in\{0,1\}} (x-p)^2 p(x) \\
&= (0-p)^2(1-p) + (1-p)^2 p \\
&= p^2(1-p) + (1-p)^2 p \\
&= p^2 - p^3 + p - 2p^2 + p^3 \\
&= p - p^2 \\
&= p(1-p)
\end{aligned}$$

3. Prove that the Bernoulli Random Variable is a special case of the Binomial Random variable for which $n = 1$. (Hint: compare pmfs.)

Solution: The pmf for a Binomial(n, p) random variable is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

with support $\{0, 1, 2 \ldots, n\}$. Setting $n = 1$ gives,

$$p(x) = p(x) = \binom{1}{x} p^x (1-p)^{1-x}$$

with support $\{0, 1\}$. Plugging in each realization in the support, and recalling that $0! = 1$, we have

$$p(0) = \frac{1!}{0!(1-0)!} p^0 (1-p)^{1-0} = 1-p$$

and

$$p(1) = \frac{1!}{1!(1-1)!} p^1 (1-p)^0 = p$$

which is exactly how we defined the Bernoulli Random Variable.

4. Suppose that X is a random variable with support $\{1, 2\}$ and Y is a random variable with support $\{0, 1\}$ where X and Y have the following joint distribution:

$$\begin{aligned}
p_{XY}(1,0) &= 0.20, \qquad p_{XY}(1,1) = 0.30 \\
p_{XY}(2,0) &= 0.25, \qquad p_{XY}(2,1) = 0.25
\end{aligned}$$

(a) Express the joint distribution in a 2×2 table.

Solution:

		X = 1	X = 2
Y	0	0.20	0.25
	1	0.30	0.25

(b) Using the table, calculate the marginal probability distributions of X and Y.

Solution:

$$\begin{aligned} p_X(1) &= p_{XY}(1,0) + p_{XY}(1,1) = 0.20 + 0.30 = 0.50 \\ p_X(2) &= p_{XY}(2,0) + p_{XY}(2,1) = 0.25 + 0.25 = 0.50 \\ p_Y(0) &= p_{XY}(1,0) + p_{XY}(2,0) = 0.20 + 0.25 = 0.45 \\ p_Y(1) &= p_{XY}(1,1) + p_{XY}(2,1) = 0.30 + 0.25 = 0.55 \end{aligned}$$

(c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.

Solution: The distribution of $Y|X = 1$ is

$$\begin{aligned} P(Y = 0|X = 1) &= \frac{p_{XY}(1,0)}{p_X(1)} = \frac{0.2}{0.5} = 0.4 \\ P(Y = 1|X = 1) &= \frac{p_{XY}(1,1)}{p_X(1)} = \frac{0.3}{0.5} = 0.6 \end{aligned}$$

while the distribution of $Y|X = 2$ is

$$\begin{aligned} P(Y = 0|X = 2) &= \frac{p_{XY}(2,0)}{p_X(2)} = \frac{0.25}{0.5} = 0.5 \\ P(Y = 1|X = 2) &= \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0.25}{0.5} = 0.5 \end{aligned}$$

(d) Calculate $E[Y|X]$.

Solution:

$$\begin{aligned} E[Y|X=1] &= 0 \times 0.4 + 1 \times 0.6 = 0.6 \\ E[Y|X=2] &= 0 \times 0.5 + 1 \times 0.5 = 0.5 \end{aligned}$$

Hence,

$$E[Y|X] = \begin{cases} 0.6 & \text{with probability } 0.5 \\ 0.5 & \text{with probability } 0.5 \end{cases}$$

since $p_X(1) = 0.5$ and $p_X(2) = 0.5$.

(e) What is $E[E[Y|X]]$?

Solution: $E[E[Y|X]] = 0.5 \times 0.6 + 0.5 \times 0.5 = 0.3 + 0.25 = 0.55$. Note that this equals the expectation of Y calculated from its marginal distribution, since $E[Y] = 0 \times 0.45 + 1 \times 0.55$. This illustrates the so-called "Law of Iterated Expectations."

(f) Calculate the covariance between X and Y using the shortcut formula.

Solution: First, from the marginal distributions, $E[X] = 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5$ and $E[Y] = 0 \cdot 0.45 + 1 \cdot 0.55 = 0.55$. Hence $E[X]E[Y] = 1.5 \cdot 0.55 = 0.825$. Second,

$$\begin{aligned} E[XY] &= (0 \cdot 1) \cdot 0.2 + (0 \cdot 2) \cdot 0.25 + (1 \cdot 1) \cdot 0.3 + (1 \cdot 2)0.25 \\ &= 0.3 + 0.5 = 0.8 \end{aligned}$$

Finally $Cov(X, Y) = E[XY] - E[X]E[Y] = 0.8 - 0.825 = -0.025$

5. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:

(a) $Cov(a + bX, c + dY) = bdCov(X, Y)$

Solution: Let $\mu_X = E[X]$ and $\mu_Y = E[Y]$. By the linearity of expectation,

$$\begin{aligned} E[a + bX] &= a + b\mu_X \\ E[c + dY] &= c + d\mu_Y \end{aligned}$$

Thus, we have

$$\begin{aligned} (a + bx) - E[a + bX] &= b(x - \mu_X) \\ (c + dy) - E[c + dY] &= d(y - \mu_Y) \end{aligned}$$

Substituting these into the formula for the covariance between two discrete random variables,

$$\begin{aligned} Cov(a+bX, c+dY) &= \sum_x \sum_y [b(x-\mu_X)]\,[d(y-\mu_Y)]\,p(x,y) \\ &= bd \sum_x \sum_y (x-\mu_X)(y-\mu_Y)p(x,y) \\ &= bdCov(X,Y) \end{aligned}$$

(b) $Corr(a+bX, c+dY) = Corr(X,Y)$ provided that b, c are positive.

Solution:

$$\begin{aligned} Corr(a+bX, c+dY) &= \frac{Cov(a+bX, c+dY)}{\sqrt{Var(a+bX)Var(c+dY)}} \\ &= \frac{bdCov(X,Y)}{\sqrt{b^2Var(X)d^2Var(Y)}} \\ &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= Corr(X,Y) \end{aligned}$$

6. Fill in the missing steps from lecture to prove the shortcut formula for covariance:

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

Solution: By the Linearity of Expectation,

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E[XY] - \mu_x E[Y] - \mu_Y E[X] + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

7. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1-\omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0,1]$, so that *negative* weights are not allowed. (This rules out short-selling.)

   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.

   Solution:

$$\begin{aligned} E[\Pi(\omega)] &= E[\omega X_1 + (1-\omega)X_2] = \omega E[X_1] + (1-\omega)E[X_2] \\ &= \omega\mu_1 + (1-\omega)\mu_2 \end{aligned}$$

   (b) If $\omega \in [0,1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.

   Solution: No. If short-selling is disallowed, the portfolio expected return must be between μ_1 and μ_2.

   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.

   Solution: $Cov(X,Y) = \rho\sigma_1\sigma_2$

   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)

   Solution:

$$\begin{aligned} Var[\Pi(\omega)] &= Var[\omega X_1 + (1-\omega)X_2] \\ &= \omega^2 Var(X_1) + (1-\omega)^2 Var(X_2) + 2\omega(1-\omega)Cov(X_1, X_2) \\ &= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho\sigma_1\sigma_2 \end{aligned}$$

   (e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

   In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio*.

Solution: The First Order Condition is:

$$2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2 + (2-4\omega)\rho\sigma_1\sigma_2 = 0$$

Dividing both sides by two and rearranging:

$$\begin{aligned}
\omega\sigma_1^2 - (1-\omega)\sigma_2^2 + (1-2\omega)\rho\sigma_1\sigma_2 &= 0\\
\omega\sigma_1^2 - \sigma_2^2 + \omega\sigma_2^2 + \rho\sigma_1\sigma_2 - 2\omega\rho\sigma_1\sigma_2 &= 0\\
\omega(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) &= \sigma_2^2 - \rho\sigma_1\sigma_2
\end{aligned}$$

So we have

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

(f) If you want a challenge, check the second order condition from part (e).

Solution: The second derivative is

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2$$

and, since $\rho = 1$ is the largest possible value for ρ,

$$2\sigma_1^2 - 2\sigma_2^2 - 4\rho\sigma_1\sigma_2 \geq 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \geq 0$$

so the second derivative is positive, indicating a minimum. This is a global minimum since the problem is quadratic in ω.